SAMPLING INSPECTION PLANS FOR CONTINUOUS PRODUCTION WHICH INSURE A PRESCRIBED LIMIT ON THE OUTGOING QUALITY

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1. Introduction. This paper discusses several plans for sampling inspection of manufactured articles which are produced by a continuous production process, the plans being designed to insure that the long-run proportion of defectives shall not exceed a prescribed limit. The plans are applicable to articles which can be classified as "defective" or "non-defective" and which are submitted for inspection either continuously or in lots. In Section 2 the notions of "average outgoing quality limit" and "local stability" are discussed. The valuable concept of average outgoing quality limit for lot inspection is due to Dodge and Romig [4], and that for inspection of continuous production to Dodge [1]. Section 3 contains a description of a simple inspection plan (SPA) applicable to to continuous production and a proof that the plan will insure a prescribed average outgoing quality limit. Section 4 contains a proof that this inspection plan also has the important property that it requires minimum inspection when the production process is in statistical control. In Section 5 is contained the description of a general class of plans which possess both these important properties.

The problem of adapting SPA to the case when the articles are submitted for inspection in lots instead of continuously, is treated in Section 6. Some methods of achieving local stability are discussed in Section 7 and a specific plan is developed there. Finally Section 8 discusses the relationship between the present work and that of the earlier and very interesting paper of H. F. Dodge [1], mentioned above.

If a quick first reading is desired the reader may omit the second half of Section 3 (which contains a proof of the fact that SPA guarantees the prescribed average outgoing quality limit) and the entire Section 4 except for its title (the proof of the statement made in the title of Section 4 occupies the whole section).

2. Fundamental notions. In this paper we shall deal only with a product whose units can be classified as "defective" or "non-defective." We shall assume that the units of the product are submitted for inspection continuously, except in Section 6, where we assume that they are submitted in lots. Throughout the paper we shall assume that the inspection process is non-destructive, that it invariably classifies correctly the units examined, and that defective units, when found, are replaced by non-defectives. By the "quality" of a sequence of units is meant the proportion of defectives in the sequence as produced. By the "outgoing quality" (OQ) of a sequence is meant the proportion of defectives after whatever inspection scheme which is in use has been applied. If this scheme involves random sampling, then in general the OQ is a chance variable.

(It depends on the variations of random sampling.) If the OQ converges to a constant p_a with probability one as the number of units produced increases indefinitely, p_a is called the "average outgoing quality" (AOQ). The AOQ when it exists is therefore the average quality, in the long run, of the production process after inspection. It is a function of both the production process and the inspection scheme. These definitions are due to Dodge [1].

The "average outgoing quality limit" (AOQL) is a number which is to depend only on the inspection scheme and not at all on the production process. Roughly speaking, it is a number, characteristic of an inspection scheme, such that no matter what the variations or eccentricities of the production process, the AOQ never exceeds it. For the purposes of this paper we shall need the following precise definition: Let c_i be zero or one according as the ith unit of the product, before application of the inspection scheme, is a non-defective or a defective, respectively. Let d_i have a similar definition *after* application of the inspection scheme. (We note that if the ith item was inspected, then $d_i = 0$; if the ith item was not inspected, then $c_i = d_i$.) The sequence $c = c_1, c_2, \cdots, c_N, \cdots$, ad inf. characterizes the production process[1]. The elements of $d = d_1, d_2, \cdots$, ad inf. are in general chance variables. The number L is called the AOQL if it is the smallest[2] number with the property that the probability is zero that

$$\limsup_{N} \frac{\sum_{i=1}^{N} d_i}{N} > L,$$

no matter what the sequence c.

It should be noted that this definition of AOQL places no restrictions whatever on the production process, since *all* sequences c are admitted. It is too much to expect a production process to remain always in control; indeed, doubt as to whether statistical control always exists may cause a manufacturer to institute an inspection scheme. The inspection schemes which we shall give below will yield a specified AOQL no matter what the variations in production are. If these schemes are employed, then, even if Maxwell's demon of gas theory fame were to transfer his activities to the production process, he would be unsuccessful in an effort to cause the AOQL to be exceeded. A dishonest manufacturer might sometimes essay to do this. If we imposed restrictions on the sequence c and

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[1] This use of an infinite sequence to describe the production process deserves a few words. What we consider in this paper are schemes applicable when the number of units produced is large and operate mathematically as if the production sequence were of infinite length. Naturally the latter is never the case in actuality. However, the larger the number of units produced the more nearly will the reality conform to the results derived from the mathematical model. While the present definition uses explicitly the notion of an infinite sequence, such a commonplace statement as "the probability is 1/2 that a coin will fall heads up" uses this notion implicitly. It is also implicit in the intuitive meaning we ascribe to such a word as "average," which is in every day use.

[2] It is not difficult to see that such a number always exists, for it is the lower bound of a set which is non-empty (it contains the point one), bounded from below (zero is a lower bound), and closed.

determined the AOQL on that basis, we would run the danger that the relative frequency of defects in the sequence of outgoing units might exceed the AOQL if it happened that the actual sequence c did not satisfy the restrictions imposed.

After we discuss below various possible sampling inspection plans which insure that the AOQL does not exceed a predetermined value L, it will be seen that for any given $L > 0$ there are many sampling inspection schemes which do this. To choose a particular sampling plan from among them the following considerations may be advanced: If two inspection plans S and S' both insure the inequality AOQL $\leq L$ and if for any sequence c the average number of inspections required by S is not greater than that required by S' and if for some sequences c the average number of inspections required by S is actually smaller than that required by S', then S may be considered, in general, a better inspection plan than S'. However, the amount of inspection required by a sampling plan is not always the *only* criterion for the selection of a proper sampling scheme. There may be also other features of a sampling plan which make it more or less desirable. We shall mention here one such feature, called "local stability," which will play a role in our discussions later. Consider the sequence d obtained from the sequence c by applying a sampling inspection scheme. Even if the AOQL does not exceed L, it may still happen that there will be many large segments of the sequence d within which the relative frequency of ones is considerably higher than L. For instance, it may happen that in the segment $(d_1, \cdots, d_m)$ the relative frequency of ones is equal to $\frac{3}{2}L$, in the segment $(d_{m+1}, \cdots, d_{2m})$ the relative frequency is equal to $\frac{1}{2}L$, in the segment $(d_{2m+1}, \cdots, d_{3m})$ the relative frequency is again equal to $\frac{3}{2}L$, and this is followed again by a segment of m elements where the relative frequency of ones is equal to $\frac{1}{2}L$, and so forth. If m is large, such a sequence d is not very desirable, since each second segment will contain too many defects. A sequence d is said to be not locally stable if there exists a large fixed integer m such that the relative frequency of ones in $(d_{k+1}, \cdots, d_{k+m})$ is considerably greater than L for *many* integral values k. On the other hand, the sequence d is said to be locally stable if for any large m the relative frequency of ones in $(d_{k+1}, \cdots, d_{k+m})$ is not substantially above L for nearly all integral values k. This is clearly not a precise definition of "local stability," but merely an intuitive indication of what we want to understand by the term, since we did not define what we mean by "large m," "many values of k," "considerably above L," etc. A precise definition of local stability will not be needed in this paper, since it is not our intention to develop a complete theory for the choice of the sampling plan. The idea of local stability will be used in this paper merely for making it plausible that some schemes we shall consider behave reasonably in this respect. A similar idea, called "protection against spotty quality," is discussed by Dodge [1]. A possible precise definition of local stability could be given in terms of the frequency with which $F(N) = \frac{1}{(k+1)} \sum_{i=N}^{N+k} d_i$ (k being fixed) lies within given limits.

3. A sampling inspection plan which insures a given AOQL no matter what the variations in the production process. The only feature of the sampling (inspection) plan (SP) studied in this section and hereafter referred to as SPA which we shall consider here is that it insures the achievement of a specified AOQL. Considerations leading to a choice among several schemes are postponed to later sections.

For convenience, let f be the reciprocal of a positive integer. SPA calls for alternating partial inspection and complete inspection. Partial inspection is performed by inspecting one element chosen at random from each of successive groups of $\frac{1}{f}$ elements. Complete inspection means the inspection of every element in the order of production. SPA is completely defined when a rule is given for ending one kind of inspection and beginning the other.

It is clear that all SP need not be of the above class. Thus, for example, a scheme might consist of partial inspection with various f's employed in various sequences. We make no attempt in this paper to examine all possible schemes. For simplicity in practical operation, alternation of complete inspection and partial inspection with fixed f would seem reasonable. The Dodge scheme [1] is of this type.

We shall also not discuss the question of a choice of the constant f, but will assume that a particular value has been chosen for various reasons and is a datum of our problem. Reasons which might influence a manufacturer in his choice of f could be contract specifications which impose a minimum on the amount of inspection, or psychological grounds to the same effect. The manufacturer may desire a certain minimum amount of inspection in order to detect malfunctioning of his production process. Also f controls local stability to some extent. The consequences of a choice of f as they appear in the theory below may also play a role.

Returning to SPA, we begin with partial inspection. Let L be the specified AOQL. Denote by k_N the number of groups of $\frac{1}{f}$ units in which defectives were found as the result of *partial* inspection from the beginning of production through the Nth unit. SPA is as follows:

(a) Begin with partial inspection.

(b) Begin full inspection whenever

$$e_N = \frac{k_N\left(\frac{1}{f} - 1\right)}{N} > L.$$

(c) Resume partial inspection when

$$e_N \leq L.$$

(d) Repeat the procedure. (It will be recalled that defective units, when found, are always to be replaced with non-defectives.)

It is to be observed that in this plan the number of partial inspections increases without limit. For, while complete inspection is going on, the value of k_N remains constant, so that after a long enough period of complete inspection the denominator N of the expression which defines e_N will have increased sufficiently for e_N to be not greater than L. On the other hand, complete inspection may never occur. This will be the case if, for example, no defectives or very few defectives are produced.

We shall now show that the AOQL of the above SP is L. We first note that, at N, e_N can increase only by $\dfrac{\left(\dfrac{1}{f} - 1\right)}{N}$. Hence, for sufficiently large N, $e_N <$ $L + \epsilon$, where $\epsilon > 0$ may be arbitrarily small.

Suppose now that the production process is subject to any variations whatsoever, i.e., the sequence

$$c = c_1, c_2, \cdots, c_N, \cdots, \text{ ad inf.}$$

is any arbitrary sequence whatever (by their definition the c_i are all zero or one). Our result is therefore proved if we show that, with probability one,

(3.1)
$$\lim_{N \to \infty} \left(e_N - \frac{1}{N} \sum_{i=1}^{N} d_i \right) = 0$$

for this arbitrary c, and that for at least one c

(3.2)
$$\lim_{N \to \infty} e_N = L.$$

Let $S(N)$ be the number of groups of $\dfrac{1}{f}$ units which have been partially inspected through the Nth unit. Define x_i as zero if in the ith partially inspected group a non-defective was found and as one if a defective was found. We have

$$k_N = \sum_{i=1}^{S(N)} x_i.$$

Since the number of times partial inspection takes place increases indefinitely, $S(N) \to \infty$ as $N \to \infty$. Also $S(N) \le fN < N$. Let α_j be the serial number of the last unit in the jth partially inspected group. Then for all j the expected value $E(x_j)$ of x_j is given by

$$E(x_j) = f\left(\sum_{i=(\alpha_j - (1/f) + 1)}^{\alpha_j} c_i \right).$$

We have, for all j

(3.3)
$$\sum_{i=(\alpha_j - (1/f) + 1)}^{\alpha_j} (c_i - d_i) = x_j$$

so that

$$E\left(\left[\frac{1}{f} - 1 \right] x_j - \sum_{\alpha_j - (1/f) + 1}^{\alpha_j} d_i \right) = 0.$$

Also from (3.3) it follows, since x_j is the value of a binomial chance variable from a population of fixed number $\left(\dfrac{1}{f}\right)$, that there exists a positive constant β such that

$$(3.4) \qquad \sigma^2\left(\left[\frac{1}{f} - 1\right]x_j - \sum_{\alpha_j-(1/f)+1}^{\alpha_j} d_i\right) < \beta$$

where $\sigma^2(x)$ is the variance of a chance variable x. Now a theorem of Kolmogoroff (Kolmogoroff [2], Fréchet [3], p. 254) states:
A sequence of chance variables with zero means and variances $\sigma_1^2,\ \sigma_2^2,\ \cdots$ converges with probability one towards zero in the sense of Cèsaro if

$$(3.5) \qquad \sum_{i=1}^{\infty} \frac{\sigma_i^2}{i^2}$$

converges. The inequality (3.4) permits us to apply this theorem to the sequence of chance variables of which the jth ($j = 1, 2, \cdots$ ad inf.) is

$$\left(\left[\frac{1}{f} - 1\right]x_j - \sum_{\alpha_j-(1/f)+1}^{\alpha_j} d_i\right),$$

since the series $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^2}$ is well known to be convergent. We therefore obtain that, with probability one,

$$\lim_{S(N)\to\infty} \frac{\left(\left[\frac{1}{f} - 1\right]\sum_{j=1}^{S(N)} x_j - \sum_{j=1}^{N} d_j\right)}{S(N)} = \lim_{N\to\infty} \frac{N}{S(N)}\left(e_N - \frac{1}{N}\sum_{i=1}^{N} d_i\right) = 0,$$

since the units which are fully inspected contribute nothing to Σd_i. Since $S(N) < N$, the desired result (3.1) is a fortiori true.

If c is such that all the c_i are one, it is readily seen that (3.2) holds. If many (this adjective can be precisely defined) defectives are produced, this will also be the case. This completes the proof of the fact that the AOQL of SPA is L no matter how capriciously the production process may vary.

4. When the production process is in statistical control, SPA requires minimum inspection. The production process is said to be in statistical control if there is a positive constant $p \le 1$ such that, for every i, the probability that $c_i = 1$ is p and is independent of the values taken by the other c's. We shall see that if the process is in statistical control and if SPA is applied to it, the specified AOQL is guaranteed with a minimum amount of inspection.

The number of units inspected through the Nth unit produced is

$$(4.1) \qquad I(N) = N - \left(\frac{1}{f} - 1\right)S(N).$$

If the process is in statistical control we have, with probability one,

$$(4.2) \qquad \lim_{N \to \infty} \frac{\sum_{i=1}^{N} c_i}{N} = p$$

by the strong law of large numbers. Shortly we shall prove the existence of a constant L^* such that, with probability one,

$$(4.3) \qquad \lim_{N \to \infty} \frac{\sum_{i=1}^{N} d_i}{N} = L^*.$$

Assume for the moment that this is so. Since it is only by inspection that defectives are removed, and the units selected for inspection are in statistical control like the original sequence, it follows that, with probability one,

$$(4.4) \qquad \lim_{N \to \infty} \frac{I(N)}{N} = \frac{1}{p}(p - L^*) = 1 - \frac{L^*}{p}$$

because, with probability one,

$$\lim_{N \to \infty} \frac{\sum_{i=1}^{N}(c_i - d_i)}{N} = p - L^*.$$

Inspection is therefore at a minimum when L^* is at a maximum compatible with the specified AOQL. By (4.3) the latter means that

$$(4.5) \qquad L^* \leq L.$$

SPA has been shown to guarantee this requirement. The optimum situation from the point of view of the amount of inspection would therefore be to have $L^* = L$, but this cannot always be achieved. The absolute minimum amount of inspection clearly is f, i.e., partial inspection exclusively. Consequently from (4.4)

$$1 - \frac{L^*}{p} \geq f$$

so that

$$(4.6) \qquad L^* \leq p(1 - f).$$

Combining (4.5) and (4.6) we see that we have to consider three cases:

Case a. If

$$(4.7) \qquad p > \frac{L}{1 - f}$$

we have to show that

$$(4.8) \qquad L = L^*.$$

Case b. If

$$(4.9) \qquad p < \frac{L}{1 - f}$$

we have to show, by (4.4), that

$$1 - \frac{L^*}{p} = f,$$

that is,

(4.10) $$L^* = p(1 - f).$$

Case c. If

(4.11) $$p = \frac{L}{1 - f}$$

we have to show that

(4.12) $$L = L^* = p(1 - f).$$

PROOF of (4.8): We have already remarked in Section 3 that in SPA partial inspection always recurs, but complete inspection need never occur. We shall show in a moment that (4.7) implies that no matter how large an integer γ is chosen, the probability of temporarily stopping partial inspection for some $N > \gamma$ is one. Assume that this is so. Choose an arbitrarily small positive ϵ, and let $\gamma > \dfrac{\left(\frac{1}{f} - 1\right)}{\epsilon}$. For a sequence where complete and partial inspection alternate infinitely many times let

$$A = \alpha_1, \; \alpha_2, \; \cdots, \; \text{ad inf.}$$

be the sequence of integers at which partial inspection ends, and let

$$B = \beta_1, \; \beta_2, \; \cdots, \; \text{ad inf.}$$

be the sequence of integers at which complete inspection ends. Then, for all j,

$$\alpha_{j+1} > \beta_j > \alpha_j.$$

From the description of SPA it follows that, for all $N > \gamma$ which belong to either A or B,

(4.13) $$| e_N - L | < \epsilon.$$

In Section 3 we proved

(3.1) $$\lim_{N \to \infty} \left(e_N - \frac{1}{N} \sum_{i=1}^{N} d_i \right) = 0$$

with probability one. Since ϵ is arbitrarily small it follows that, with probability one,

(4.14) $$\lim_{\substack{N \to \infty \\ (N \text{ in } A \text{ or } B)}} \frac{\sum_{i=1}^{N} d_i}{N} = L.$$

To complete the proof of (4.8) we have still to show that L^* exists and that the probability is one that complete inspection will occur infinitely many times. First we prove that L^* exists.

As N increases during an interval of complete inspection, $D(N) = \sum_{i=1}^{N} d_i$ remains constant. Hence $\dfrac{D(N)}{N}$ decreases monotonically. Since for the ends of such intervals (4.14) holds, it follows that (4.14) holds as $N \rightarrow \infty$ and is a member of A, B, or an interval (α_j, β_j) for all j.

Let $N \rightarrow \infty$ while always being in the interior of an interval $(\beta_j, \alpha_{j+1}]$, $j = 1, 2, \cdots$, ad inf., which contains α_{j+1} but not β_j. Let N^* be the total number of units in these intervals through the Nth unit produced. Let N_1 and N_2 be such that

$$\beta_j = N_1 < N_2 < \alpha_{j+1}.$$

Then

$$N_2^* - N_1^* = N_2 - N_1.$$

Since the production process is in statistical control, we have, by the strong law of large numbers,

$$(4.15) \qquad \lim_{N \to \infty} \frac{D(N)}{N^*} = p(1 - f) = p'$$

with probability one. Let δ^* be the general designation for numbers $< \epsilon$ in absolute value, so that all δ^* are not the same. With probability one for almost all N, we have by (4.15)

$$\frac{D(N_1)}{N_1^*} = p' + \delta^*$$

$$\frac{D(N_2)}{N_2^*} = p' + \delta^*.$$

Write

$$\frac{[D(N_2) - D(N_1)]}{(N_2 - N_1)} = K.$$

Now

$$\frac{D(N_2)}{N_2^*} = \frac{D(N_1) + [D(N_2) - D(N_1)]}{N_1^* + (N_2^* - N_1^*)} = \frac{D(N_1) + [D(N_2) - D(N_1)]}{N_1^* + (N_2 - N_1)}$$

$$= \frac{(p' + \delta^*)N_1^* + K(N_2 - N_1)}{N_1^* + (N_2 - N_1)} = p' + \delta^*.$$

Hence

$$(4.16) \qquad K(N_2 - N_1) = 2\delta^* N_1^* + (p' + \delta^*)(N_2 - N_1).$$

Now suppose (4.3) does not hold. From the definition of AOQL it follows that for some $\eta > \epsilon$ there exist sequences (whose totality has a positive probability) so that, for infinitely many N_2 we have

$$(4.17) \qquad \frac{D(N_2)}{N_2} = \frac{D(N_1) + [D(N_2) - D(N_1)]}{N_1 + (N_2 - N_1)} < L - 4\eta.$$

For large enough N_1, from (4.14),

$$\frac{D(N_1)}{N_1} = L + \delta*$$

with probability one and hence, using (4.16) in (4.17)

$$(4.18) \qquad N_1(L + \delta*) + 2\delta*N_1^* + (p' + \delta*)(N_2 - N_1)$$
$$< LN_1 + L(N_2 - N_1) - 4\eta N_2$$

from which, using the fact that $p' \geq L$ (from (4.7)), we get

$$(4.19) \qquad N_1\delta* + 2N_1^*\delta* + \delta*(N_2 - N_1) < -4\eta N_2 .$$

((4.18) and (4.19) hold for the sequences for which (4.17) holds, except perhaps on a set of sequences whose probability is zero.) Since $N_1^* \leq N_1$ and $|\delta*| < \eta$, we have, on the other hand,

$$(4.20) \qquad N_1\delta* + 2N_1^*\delta* + \delta*(N_2 - N_1) \geq -3\eta N_1 - \eta(N_2 - N_1)$$
$$> -4\eta N_1 - 4\eta(N_2 - N_1) = -4\eta N_2$$

which contradicts (4.19) and proves the desired result ((4.3) and (4.8)), except that it remains to prove that, no matter how large γ, the probability of temporarily stopping partial inspection at some $N > \gamma$ is one. Let $\gamma_0 \geq \gamma$ be some integer at which partial inspection is going on. From (4.2) and (4.7) it would follow, if partial inspection never ceased on a set of sequences with positive probability, that, on this set, with conditional probability one, for N sufficiently large and ϵ sufficiently small,

$$\frac{k_N - k_{\gamma_0}}{f(N - \gamma_0)} > \frac{L}{1 - f} + \epsilon,$$

$$\frac{N}{N - \gamma_0} \frac{k_N(1 - f)}{fN} > L + (1 - f)\epsilon,$$

$$e_N > L \frac{N - \gamma_0}{N} + \frac{(N - \gamma_0)(1 - f)\epsilon}{N},$$

$$e_N > L + \frac{(1 - f)\epsilon}{2}.$$

This contradiction proves that complete inspection is eventually resumed and completes the proof of minimum inspection in Case a.

PROOF of (4.10): We shall prove that (4.9) implies that, with probability one, complete inspection will cease, never to be resumed. For, from (4.15) and (4.9) it follows that for N sufficiently large and ϵ sufficiently small,

$$(4.21) \qquad \frac{D(N)}{N^*} = p' + \delta^* < L - 2\epsilon.$$

Hence, a fortiori,

$$(4.22) \qquad \frac{D(N)}{N} < L - 2\epsilon.$$

((4.21) and (4.22) hold with probability one.)
(3.1) states that, with probability one,

$$\lim_{N \to \infty} \left(e_N - \frac{D(N)}{N} \right) = 0.$$

Hence for all N sufficiently large, with probability one,

$$e_N < L - \epsilon,$$

i.e., with probability one complete inspection is never resumed.

When (4.9) holds, therefore, with probability one and with a finite number of exceptions SPA will require only partial inspection.

PROOF of (4.12): If $p = \dfrac{L}{1-f}$ and complete inspection finally never resumes, then (4.12) follows easily. If $p = \dfrac{L}{1-f}$ and partial and complete inspection alternate infinitely many times, then the proof is similar to that of (4.8) and is therefore omitted. In either case the desired result follows.

5. A class of SP all of which insure both a given AOQL and minimum inspection.
Let the definition of SPA be modified in the following particulars:

(b) Begin full inspection whenever

$$e_N = \frac{k_N\left(\frac{1}{f} - 1\right)}{N} > L + \phi(N).$$

(c) Resume partial inspection when

$$e_N \leq L - \psi(N).$$

Let $\phi(N)$ and $\psi(N)$ be such that

$$-\psi(N) \leq \phi(N)$$

$$\lim_{N \to \infty} \phi(N) = \lim_{N \to \infty} \psi(N) = 0.$$

(SPA corresponds to the case $\phi(N) \equiv \psi(N) \equiv 0$.) Then all the SP of this class have the property that the AOQL is L and that inspection is at a minimum in

the sense of Section 4. The proofs are essentially the same as those for SPA and hence will be omitted.

6. The inspection plans of Section 5 can also be applied to lot inspection. We shall carry on the discussion of this section in terms of SPA, but the results apply to all the members of the class of plans described in Section 5. We shall show that SPA can also be applied when the product is submitted for inspection in lots. Although we assumed previously that the units of the product are arranged in order of production, the results obtained for SPA remain valid for any arbitrary arrangement of the units. If the product is submitted in lots we may arrange the units as follows: Let l_1, l_2, $\cdots$, etc. be the successive lots in the order of their submission for inspection. Within each lot we consider the units arranged in the order in which they are chosen for inspection. In this way we have arranged all units in an ordered sequence and the inspection can be applied as described before. Thus, we start with partial inspection, i.e., we take out groups of $\frac{1}{f}$ elements in l_1 and inspect one unit (selected at random) from each of these groups. When $e_N > L$, we start complete inspection and revert to partial inspection as soon as $e_N \leq L$. When the units in l_1 are used up in the process of inspection, we continue, using the units of l_2, etc.

If it is found inconvenient to take out a group of $\frac{1}{f}$ units and then to select one unit for inspection, we could modify the sampling inspection plan as follows: Instead of taking out a group of $\frac{1}{f}$ units and then selecting at random one unit from it, we select at random *one* unit from the uninspected part of the lot and look upon this unit as the unit selected at random from a hypothetical group of $\frac{1}{f}$ units. Thus we can proceed exactly as before, except that we have to keep in mind that with each unit inspected under "partial inspection" we have used up another set of $\frac{1}{f} - 1$ units. Thus, as soon as $\left(\frac{1}{f} - 1\right)$ times the number of units inspected under "partial inspection" becomes equal to or greater than the number of units in the uninspected part of the lot, the inspection of that lot is already terminated, and we have to start using the units of the next lot. The inconvenience caused by the necessity of keeping track of the number of units inspected under "partial inspection" and of the number of units in the uninspected part of the lot can be eliminated by further modifying the inspection plan as follows: Instead of beginning complete inspection as soon as $e_N > L$, we continue "partial inspection" until $E_N = e_N - L$ is so large that complete inspection of all the units of the lot not yet used up has to be made in order to bring e_N down to L at the end of the lot. This leads to the following sampling procedure, to be known as SPB: Let N_0 be the number of units in the lot, let N_L be the serial number of the last unit in the preceding lot, and let $E(N_L) =$

$N_L E_{N_L} = N_L(e_{N_L} - L)$ be the "excess" carried over from the preceding lot. For simplicity assume that the following are all integers:

$$LN_0 = M$$

$$\frac{fM}{1 - f} = M^*$$

$$fN_0 = N^*$$

and

$$\frac{fE(N_L)}{1 - f} = E^*.$$

The inspection procedure is then as follows: Inspect successive units drawn at random until either

(a) $M^* - E^*$ defectives have been found in the first $N' < N^*$ units inspected. In this case inspect further an additional $N_0 - \dfrac{N'}{f}$ units and this terminates the inspection of the lot. The excess to be carried over to the next lot is then zero. Or

(b) N^* units have been inspected and the number of defectives found is $H \leq M^* - E^*$. In this case the inspection of the lot is terminated and the present negative excess

$$E(N_L + N_0) = [H - (M^* - E^*)]\frac{(1 - f)}{f}$$

is carried over to the next lot. (The serial number of the last element in the present lot is $N_L + N_0$ and

$$e_{(N_L+N_0)} = \frac{N_L e_{N_L} + H\dfrac{(1 - f)}{f}}{N_L + N_0} .$$

Hence the present excess is

$$(N_L + N_0)[e_{(N_L+N_0)} - L] = N_L e_{N_L} + H\frac{(1 - f)}{f} - LN_L - LN_0$$

$$= N_L(e_{N_L} - L) + H\frac{(1 - f)}{f} - M$$

$$= \frac{(1 - f)}{f}[H - M^* + E^*],$$

as given above.)

We note an important property of SPB: The excess carried over from a preceding lot is never positive.

7. Possible modifications of the SP to achieve local stability. Although the sampling plans discussed in previous sections are optimum in the sense that they guarantee the desired AOQL with a minimum of inspection when the production process is in statistical control, they do not always behave very favorably as far as local stability is concerned. To make this point clear, consider the following example: Suppose that during a very long initial time period the production process functions very well and the relative frequency of defectives produced is well below L. Thus, applying SPA, say, $e_N - L$ will be considerably less than zero at the end of this period. Now suppose that then the production process suddenly deteriorates and the number of defectives produced during the next period of time is considerably higher than L. In spite of that, complete inspection will not begin for quite some time because e_N became so small during the initial period. Thus there will be a long segment in the sequence of outgoing units within which the relative frequency of defectives will be larger than the prescribed AOQL. Of course, this segment will be counterbalanced by other segments where the relative frequency of defectives will be below the AOQL, so that the AOQL will not be violated. Nevertheless, the occurrence of long segments with too many defectives, i.e., a lack of local stability, is not desirable.

It should be noted that, even though SPA was not designed to achieve considerable local stability, drastic lack of local stability cannot occur when the production process is in statistical control and SPA is employed. In the example given above where the outgoing quality was not locally stable, it was assumed that there were variations in the production process. The existence of statistical control acts as an important stabilizing factor on the quality.

In this section we want to discuss several possible modifications of SPA which will insure a greater degree of local stability. One such modification is the following: We choose a positive constant A and we define the excess E_N^* for each value N as follows: $E^*(N)$ is equal to the excess $E(N)$ as originally defined ($= N[e_N - L]$) as long as for all $N' \leq N$, $E(N') \geq -A$. The difference $E^*(N + 1) - E^*(N) = E(N + 1) - E(N)$ for all N for which $E(N + 1) - E(N) \geq 0$. If $E(N + 1) - E(N) < 0$, then $E^*(N + 1) = \max [E^*(N) + \{E(N + 1) - E(N)\}, -A]$. In other words, with this modification of the sampling inspection plan we set a lower bound $-A$ for the excess. When the excess is positive we begin complete inspection, and revert to partial inspection when the excess becomes non-positive. The effect of this is that, if the proportion of defectives produced becomes large, complete inspection will not be delayed very long, although the proportion of defectives produced in the preceding period may have been considerably below L. It is clear that this modification of SPA does not increase the AOQL. However, the amount of inspection will be somewhat increased, especially when the quality of the product is less than or only slightly greater than L. If the constant A is large, the increase in the amount of inspection is only slight, but also the degree of local stability achieved is not very high. On the other hand, if A is small, the increase

in the amount of inspection may be considerable, but a high degree of local stability is achieved. Thus, the choice of A should be made so that a proper balance between local stability and amount of inspection is achieved.

Modifying SPA by setting a lower limit for the excess has the disadvantage that the mathematical treatment of this case is involved. We shall, therefore, consider another modification of the inspection plan which will have largely the same effect, but whose mathematical treatment appears to be much simpler. A fixed positive integer N_0 is chosen and the inspection scheme is designed so that $E_{N_0} \leq 0$ is assured. If E_{N_0} is negative, we replace it by zero. In other words, no excess is carried over from the first segment of N_0 units to the next segment of N_0 units. Thus, the second segment of N_0 units is treated exactly the same way as if it were the first segment, and this is repeated for each consecutive segment of N_0 units. This modification of SPA (the resulting plan is to be known as SPC) has essentially the same effect as setting a lower bound for the excess. Again it is clear that by this modification the AOQL is not increased, but the amount of inspection may be increased. The latter is particularly true when N_0 is small, which corresponds to very high local stability requirements. More efficient plans than SPC can probably be devised for this situation.

Undoubtedly, there are many other possible modifications of the inspection plan by which a greater degree of local stability can be achieved at the price of somewhat increased inspection. It is not the purpose of this paper to enumerate all these possibilities or to develop a theory as to which of them may be considered an optimum procedure. We shall restrict ourselves to a discussion of the mathematical consequences of SPC. First we define it precisely. If it is to be applied to inspection of lots of size N_0 then SPC is simply SPB with $E(N_L)$ and E^* always zero. When applied to continuous production it will operate as follows: Assume for convenience that $M = LN_0$, $N^* = fN_0$, and $\dfrac{fM}{1-f} = M^*$ are all integers.

(a) Begin each segment of N_0 units with partial inspection, i.e., inspect one unit chosen at random from each successive group of $\dfrac{1}{f}$ units. Continue partial inspection until one of the following events occurs: either

(b) M^* defectives are found. In this case begin complete inspection with the first unit which follows the group in which the last of the M^* defectives was found and continue until the end of the segment of N_0 units.
or

(b′) N^* groups of $\dfrac{1}{f}$ units are partially inspected.

(c) Repeat with the next segment of N_0 units.

Comparison with SPB shows that, in SPC, if (b) occurs earlier or at the same time as (b′), then $E_{N_0} = 0$, while if (b′) occurs before (b) we have $E_{N_0} < 0$. In contradistinction to SPB, in SPC there is no carrying over of the excess.

Let us determine the AOQ for SPC when the production process is in a state

of statistical control. Denote by p the probability that a unit produced will be defective. Let the chance variable H denote the number of defectives found during partial inspection. The probability that $H = i < M^*$ is

$$\binom{N^*}{i} p^i (1 - p)^{N^* - i}.$$

$H \leq M^*$ always. We have, when $H = i$,

$$E(N_0) = \frac{(1 - f)i}{f} - LN_0,$$

and hence

$$N_0 e_{N_0} = \frac{(1 - f)i}{f}.$$

The AOQ is therefore $\dfrac{(1 - f)}{fN_0}$ multiplied by the expected value of H and is therefore

$$(7.1) \quad \frac{(1 - f)}{fN_0} \left[M^* - \sum_{i=0}^{M^*-1} (M^* - i) \binom{N^*}{i} p^i (1 - p)^{N^* - i} \right]$$

$$= L \left[1 - \frac{1}{M^*} \sum_{i=0}^{M^*-1} (M^* - i) \binom{N^*}{i} p^i (1 - p)^{N^* - i} \right].$$

The reduction from the original quality p to the AOQ was achieved by inspecting a fraction of units which is $\dfrac{1}{p}$ times the reduction in the frequency of defectives. Hence, with probability one, the fraction of units inspected when the production process is in statistical control is

$$(7.2) \quad I = 1 - \frac{L}{p} + \frac{(1 - f)}{pN^*} \sum_{i=0}^{M^*-1} (M^* - i) \binom{N^*}{i} p^i (1 - p)^{N^* - i}.$$

When $p \geq \dfrac{L}{1 - f}$ we see from Section 4 that the third term of the right member of (7.2) represents the price paid in fraction of inspection above the minimum in return for the local stability achieved. When $p < \dfrac{L}{1 - f}$ the additional inspection is of course $I - f$.

As N_0 becomes larger, SPC becomes more and more like SPA, and consequently the amount of inspection tends to the minimum. As N_0 becomes smaller, the degree of local stability achieved becomes higher and must be paid for by an increasing amount of inspection. An illustrative example will be given in the next section. It has already been pointed out that the mere existence of statistical control implies a considerable amount of local stability even when SPA is applied.

The only practical difficulty which may arise in evaluating the formulas in (7.1) and (7.2) might come from attempting to evaluate

$$T' = \sum_{i=0}^{M^*-1} (M^* - i) \binom{N^*}{i} p^i (1 - p)^{N^*-i}.$$

For those values of the parameters which are likely to occur in application, a good approximation to T' (exactly how good we shall not investigate here) is given by

$$T = \sum_{i=0}^{M^*-1} (M^* - i) \frac{e^{-N^*p}(N^*p)^i}{i!}.$$

A table of T for integral values of M^* from 2 to 16 and for integral values of N^*p from 1 to 25 is given below. The computations were performed under the direction of Mr. Mortimer Spiegelman of the Metropolitan Life Insurance Company, to whom the authors are deeply obliged.

$$\textit{Table of } T = \sum_{i=0}^{M^*-1} (M^* - i) \frac{e^{-N^*p}(N^*p)^i}{i!}$$

$M^* - 1$	N^*p											
	1	2	3	4	5	6	7	8	9	10	11	12
1	1.10	.54	.25	.11	.05	.02	.01	.00	.00	.00	.00	.00
2	2.02	1.22	.67	.35	.17	.08	.04	.02	.01	.00	.00	.00
3	3.00	2.08	1.32	.78	.44	.23	.12	.06	.03	.01	.01	.00
4	4.00	3.02	2.13	1.41	.88	.52	.29	.16	.08	.04	.02	.01
5	5.00	4.01	3.05	2.20	1.49	.96	.59	.35	.20	.11	.06	.03
6	6.00	5.00	4.02	3.08	2.26	1.57	1.04	.66	.41	.24	.14	.08
7	7.00	6.00	5.01	4.03	3.12	2.31	1.64	1.12	.73	.46	.28	.17
8	8.00	7.00	6.00	5.01	4.05	3.16	2.37	1.71	1.19	.79	.51	.32
9	9.00	8.00	7.00	6.00	5.02	4.08	3.20	2.43	1.77	1.25	.85	.56
10	10.00	9.00	8.00	7.00	6.01	5.03	4.10	3.24	2.48	1.83	1.31	.91
11	11.00	10.00	9.00	8.00	7.00	6.01	5.05	4.13	3.28	2.53	1.89	1.37
12	12.00	11.00	10.00	9.00	8.00	7.01	6.02	5.07	4.16	3.32	2.58	1.95
13	13.00	12.00	11.00	10.00	9.00	8.00	7.01	6.03	5.08	4.19	3.36	2.63
14	14.00	13.00	12.00	11.00	10.00	9.00	8.00	7.01	6.04	5.10	4.22	3.40
15	15.00	14.00	13.00	12.00	11.00	10.00	9.00	8.01	7.02	6.05	5.12	4.25

8. The SP of H. F. Dodge. H. F. Dodge [1] has proposed a very interesting SP for continuous production. The plan is defined by two constants i and f and may be described as follows: Begin with complete inspection of the units consecutively as produced and continue such inspection until i units in succession are found non-defective. Thereafter inspect a fraction f of the units. Continue partial inspection until a defect is found. Then start complete inspection again and continue until i units in succession are found non-defective. Repeat the procedure.

Dodge [1] derived formulas for determining the AOQL corresponding to any

pair i and f, under the assumption that the production process is in a state of statistical control. Dodge's formulas for the AOQL are not necessarily valid if we do not make this restriction on the production process, i.e., if we admit that the probability p that a unit will be defective may vary in any arbitrary way during the production process. This, of course, is not a criticism of the derivation of the formulas; it cannot be considered surprising that a formula is not valid under assumptions different from those under which it was derived. However, it is relevant to point out the fact that the Dodge SP does not guarantee the AOQL under all circumstances, so that care must be taken to ensure that certain requirements are met. Exactly what these requirements are is not known; statistical control is a sufficient condition, but is probably not necessary and could be weakened. It seems likely to the authors that, if p varies only slowly (with N) with infrequent "jumps," the Dodge SP will produce results which will exceed the AOQL by little, if at all. But if the "jumps" are numer-

$$\text{Table of } T = \sum_{i=0}^{M^*-1} (M^* - i) \frac{e^{-N^* p}(N^* p)^i}{i!}$$

(Continued)

$M^* - 1$	$N^* p$												
	13	14	15	16	17	18	19	20	21	22	23	24	25
1	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
2	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
3	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
4	.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
5	.02	.01	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00
6	.04	.02	.01	.01	.00	.00	.00	.00	.00	.00	.00	.00	.00
7	.10	.05	.03	.02	.01	.00	.00	.00	.00	.00	.00	.00	.00
8	.20	.12	.07	.04	.02	.01	.01	.00	.00	.00	.00	.00	.00
9	.36	.23	.14	.08	.05	.03	.01	.01	.00	.00	.00	.00	.00
10	.61	.40	.26	.16	.10	.06	.03	.02	.01	.01	.00	.00	.00
11	.97	.66	.44	.29	.18	.11	.07	.04	.02	.01	.01	.00	.00
12	1.43	1.02	.71	.48	.32	.20	.13	.08	.05	.03	.02	.01	.01
13	2.00	1.48	1.07	.75	.52	.35	.23	.15	.09	.06	.03	.02	.01
14	2.68	2.05	1.54	1.12	.80	.55	.38	.25	.16	.10	.07	.04	.02
15	3.44	2.72	2.10	1.59	1.17	.84	.59	.41	.27	.18	.12	.07	.05

ous and appropriately spaced it is possible to exceed the AOQL by substantial amounts, as the example below will show. The Dodge plan was intended to serve as an aid to the detection and correction of malfunctioning of the production process and this use would tend to prevent the occurrence of such a phenomenon. Parenthetically, it should be remarked that the information obtained in the course of inspection according to either the plans discussed in this paper or any reasonable scheme should, if possible, be sent at once to the producing divisions for their guidance.

An example to show that the AOQL can be exceeded can be constructed as

follows: Let $i = 54$ and $f = 0.1$. Then according to the graphs of [1], page 272, the AOQL should be 0.02. Define a sequence of 60 successive units free of defectives as a segment of type 1, and a sequence of 60 successive units where the production process is in statistical control with $p = 0.1$, as a segment of type 2. Suppose that the sequence of units produced consists of segments of types 1 and 2 always alternating. Then it follows that the first item inspected in a segment of type 2 is always inspected on a partial inspection basis. We now assume that, unless the occurrence of a defective has previously terminated partial inspection, the 1st, 11th, 21st, 31st, 41st, and 51st items in a segment of type 2 will be chosen for partial inspection, and if the 1st item is found defective, the entire segment of type 2 will be cleared of defectives. (Both of these assumptions favor the Dodge SP.) Then the situation is as described in the following table:

	(1) *Probability of first terminating partial inspection at each item*	(2) *Expected number of defectives remaining in segment of type 2 after partial inspection has been terminated*	(3) (1) x (2)
1st	.1	0	0
11th	$(.9)(.1) = .09$	.9	.081
21st	$(.9)^2(.1) = .081$	1.8	.1458
31st	$(.9)^3(.1) = .0729$	2.7	.19683
41st	$(.9)^4(.1) = .06561$	3.6	.236196
51st	$(.9)^5(.1) = .059049$	4.5	.2657205

Probability that an entire segment of type 2 will be partially inspected	*Expected number of defectives left in a segment of type 2 which has been inspected only partially*	*Product*
$(.9)^6 = .531441$	5.4	2.8697814

Sum $= 3.7953279$

The AOQ is therefore $\dfrac{3.7953279}{120} = .0316+$, while $L = .02$.

It is therefore difficult to compare the Dodge plan with any of the plans described in this paper with respect to their effect on a production process not in statistical control. If the production process is in statistical control, then, as we have already seen, SPA requires minimum inspection (and, incidentally, because of the existence of statistical control, produces a fair degree of local stability). If, when statistical control exists, one requires both maintenance of a given AOQL and a higher degree of local stability than is produced by SPA, the relevant comparison is between the Dodge plan and SPC. Both will probably give good results as regards local stability, but it is not possible at present to make

these intuitive notions precise, as we have not given an exact definition of local stability. The following example (in which statistical control is assumed) may not be unrepresentative of what the situation is with regard to the amount of inspection required.

Fraction of product inspected under the Dodge plan and under SPC when
$L = .045$ $f = .1$

p	Fraction of product inspected under the Dodge plan	Fraction of product inspected under SPC when		
		$N_0 = 400$	$N_0 = 1000$	$N_0 = 2000$
.01	.12	.12	.10	.10
.02	.15	.17	.11	.10
.03	.19	.22	.14	.11
.04	.23	.28	.19	.15
.05	.28	.34	.26	.21
.06	.33	.40	.33	.29
.07	.39	.45	.39	.37
.08	.45	.50	.46	.44
.09	.52	.54	.51	.50
.10	.58	.57	.55	.55

The decrease in inspection required by SPC as N_0 increases is evident in this table. When $N_0 = 2000$ SPC requires less inspection than the Dodge plan, when $N_0 = 400$ it requires more inspection than the Dodge plan. How the various degrees of local stability achieved compare remains an open question. The case when $N_0 = 400$ probably lies in the region where SPC is inefficient (as regards amount of inspection) and corresponds to a high degree of local stability.

We note that both plans call for increased inspection as the quality worsens (p increases). If the manufacturer is required to pay for the inspection this serves as an added incentive to improve quality of output.

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