ON THE DESIGN OF EXPERIMENTS FOR WEIGHING AND MAKING OTHER TYPES OF MEASUREMENTS

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1. Introduction. In a recent paper, Hotelling [1] has discussed the basic principles of the theory of the design of efficient experiments for estimating the true unknown weights of p given objects by means of a specified number N of weighings, $p \leq N$ in case the scale is free from bias and $p \leq N - 1$ if it has a bias the unknown value of which has to be estimated from the same data. He has emphasized the importance of these designs in other kinds of measurements besides weighing of objects and has called attention to the need for further mathematical research for obtaining a "comprehensive general solution." Such a solution has now been obtained in case the number of weighings N is at our choice. Some other general designs have also been given in this paper for specified values of N and p.

2. Estimation of unknown weights and efficiency of a design. Using Hotelling's notation, we may write

$$E(y_\alpha) = \sum_{i=1}^{p} x_{i\alpha} b_i \tag{1}$$

where $i = 1, 2, \cdots p$, on the assumption that there is either zero bias in the scale or the bias is known *a priori*, and $\alpha = 1, 2, \cdots N$. $E(y_\alpha)$ is the expectation of the αth weighing. For a biassed scale, we may take $i = 0, 1, 2, \cdots p$. The efficient estimate of each of the b_i's has been derived by Hotelling by the method of least squares. It is of interest to obtain these estimates by the use of the theory of linear estimation as developed by Bose [2] and Rao [3].

Assuming that $y_1, y_2, \cdots y_N$ are N stochastic variates forming a multivariate normal system with the variance and covariance matrix given by

$$u = [u_{ij}], \tag{2}$$

it follows from Rao's generalization of Markoff's theorem that the best unbiassed estimates of the b_i's are given by the solutions of the normal equations

$$X'U^{-1}XB' = X'U^{-1}Y', \tag{3}$$

where $B = [b_1 b_2 \cdots b_p]$ and $Y = [y_1 y_2 \cdots y_N]$, and B' and Y' denote as usual the transpose of the row vectors B and Y, i.e. column vectors.

In the present case, the assumption is that all the N stochastic variates are uncorrelated and have a common variance σ^2, so that

$$U^{-1} = \frac{1}{\sigma^2} I . \tag{4}$$

Hence the normal equations in (3) reduce to

(5) $$X'XB' = X'Y',$$

which are exactly the same as the normal equations given by Hotelling, since

(6) $$X'X = [a_{ij}]$$

where $a_{ij} = S(x_{i\alpha}x_{j\alpha})$

Let $C = [c_{ij}]$ denote the reciprocal of the matrix $X'X$, so that $V(b_i) = c_{ii}\sigma^2$ and cov $(b_ib_j) = c_{ij}\sigma^2$. Then the mean variance of the p unknowns for a design is given by

(7) $$v_m = \frac{\sigma^2}{N} \cdot \frac{N \sum_{i=1}^{p} c_{ii}}{p}.$$

If the main object of the experiment is to estimate the unknowns with the least variance, the most efficient design (for a specified value of N) would be the one for which the *minimum minimorum* of σ^2/N is attained for all the p unknowns so that the mean variance in this case is σ^2/N. The factor, $N \sum_{i=1}^{p} c_{ii}/p$, on the right-hand side of (7), therefore, measures the increase in variance resulting from the adoption of any design other than the most efficient design. Its reciprocal, $\frac{p}{N \sum_{i=1}^{p} c_{ii}}$, may appropriately be defined as the *efficiency* of a given design for providing estimates of the p unknowns. This quantity will now be utilized for judging the relative precision of the general designs discussed in the subsequent paragraphs.

3. Design for $N = 2^m$, $p \leq 2^m$ (zero bias) or $p \leq 2^m - 1$ (non-zero bias). By utilizing the properties of a 2-sided m-fold completely orthogonalized Hyper-Graeco-Latin hyper-cube of the *first* order introduced by the author [4], it is easy to see that for $N = 2^m$, $p \leq 2^m$ (when there is zero bias) or $p \leq 2^m - 1$ (when there is bias), m being any positive integer, a completely orthogonalized design can be constructed with each unknown weight estimated with the minimum variance σ^2/N. As remarked by Hotelling in the case of $N = 4$, $p = 4$ (for zero bias) or $p = 3$ (if there is bias), the matrix $X'X$ for this design is a scalar matrix of order $p \times p$ if there is zeró bias, or of order $(p + 1) \times (p + 1)$ if there is bias, each of the diagonal elements being N. The reciprocal matrix is also a scalar matrix in which each of the diagonal elements is $1/N$ so that the estimates of all the unknowns are mutually orthogonal.

As a particular case of this general design, we may take $N = 16$, $p = 16$ (for zero bias) or $p = 15$ (if there is bias), the completely orthogonalized design for which is represented by the matrix

(8)
$$X = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\
1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \\
1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 \\
1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1
\end{pmatrix}$$

for which $X'X$ is a scalar matrix of order 16×16, each diagonal element being 16. Again, a completely orthogonalized design for $N = 16$, $p < 16$ (for zero bias) or $p < 15$ (if there is bias) is represented by a matrix X obtained from the matrix in (8) by omitting any $16 - p$ of its columns if there is zero bias, or $16 - p - 1$ of its columns if there is bias. In the matrix X, permutation of rows and columns is permissible and each such matrix represents a completely orthogonalized design.

For the design given by Hotelling[1] for $N = 4$, $p = 3$ (zero bias), the efficiency is 35 per cent. The completely orthogonalized design for which the efficiency is 100 per cent is represented by the matrix

(9)
$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

4. First design for $N = 2^m + 1$, $p \leq 2^m$ (zero bias) or $p \leq 2^m - 1$ (non-zero bias). For $N = 2^m + 1$, $p \leq 2^m$ (zero bias) or $p \leq 2^m - 1$ (if there is bias), m being any positive integer, probably the most efficient design available seems to be that represented by the matrix X obtained from the corresponding matrix

[1] The allusions here and at the end of the next section are to designs on p. 305 of the Hotelling paper [1], a passage concerned with designs subject to the restriction that the entries on the matrix be 0's and +1's only, as is necessary in many types of measurement. The more efficient designs given above, whose matrices involve −1's also, can be used only in such cases as that of weighing in a balance, where the objects under investigation can be put, some in one pan and some in the other. Such situations are considered in a different part of Hotelling's paper.

X for the general design of Section 3 above by adding a row 1, 1, $\cdots$ 1 to it. The matrix $X'X$ for this design then comes out as

$$X'X = \begin{pmatrix} N & 1 & 1 \cdots & 1 \\ 1 & N & 1 \cdots & 1 \\ 1 & 1 & N \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 \cdots & N \end{pmatrix}, \tag{10}$$

which is a symmetrical matrix of order $p \times p$ if there is zero bias, or of order $(p + 1) \times (p + 1)$ if there is bias. The variance of each unknown for this design is

$$\frac{\sigma^2}{N - \dfrac{p-1}{N+p-2}} \quad \text{for zero bias,} \tag{11}$$

or

$$\frac{\sigma^2}{N - \dfrac{p}{N+p-1}} \quad \text{if there is bias.} \tag{12}$$

Thus the efficiency of this design is

$$1 - \frac{p-1}{N(N+p-2)} \quad \text{for zero bias,} \tag{13}$$

or

$$1 - \frac{p}{N(N+p-1)} \quad \text{if there is bias.} \tag{14}$$

The loss of efficiency resulting from the adoption of this design is, therefore, $\frac{p-1}{N(N+p-2)}$ for zero bias or $\frac{p}{N(N+p-1)}$ if there is bias.

As a particular case of this, for $N = 5, p = 2$ (zero bias), probably the most efficient design available is specified by

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix}. \tag{15}$$

The variance of each unknown in this case is $\frac{5\sigma^2}{24}$ and the efficiency of the design is 96 per cent. For the design given by Hotelling for this case, the variance of each unknown is $\frac{4\sigma^2}{7}$ and the efficiency is 35 per cent. It would thus appear that, as judged by the criterion of efficiency as defined here, the design represented by the matrix in (15) is more efficient than Hotelling's design.

5. Second design for $N = 2^m + 1$, $p \leq 2^m$ (zero bias) or $p \leq 2^m - 1$ (non-zero bias). Another interesting design for these values of N and p is that represented by the matrix X obtained by adding a row 1, 0, $\cdots$ 0 to the corresponding matrix X for the general design in Section 3 above. The matrix $X'X$ for this design is then the diagonal matrix

$$X'X = \begin{pmatrix} N & 0 & \cdots & 0 \\ 0 & N-1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & N-1 \end{pmatrix} \tag{16}$$

of order $p \times p$ (for zero bias) or $(p + 1) \times (p + 1)$ (for non-zero bias). As the reciprocal of this matrix is also a diagonal matrix, the estimates of all the unknowns are mutually orthogonal. The efficiency of this design is

$$\frac{(N-1)p}{Np - 1} \quad \text{for zero bias,} \tag{17}$$

or

$$\frac{N-1}{N} \quad \text{for non-zero bias.} \tag{18}$$

By comparing the efficiency of the first design given in (13) and (14) with that of the second design in (17) and (18) respectively, it would appear that the efficiency of the first design is always higher than that of the second design for non-zero bias, and is also higher in the case of zero bias for $p > 1$, but equal for $p = 1$.

6. First design for $N = 2^m + r$, $p \leq 2^m$ (for zero bias) or $p \leq 2^m - 1$ (for non-zero bias). For $N = 2^m + r$, $p \leq 2^m$ (for zero bias) or $p \leq 2^m - 1$ (for non-zero bias), m being any positive integer and r any positive integer $< 2^m$, a highly efficient design is represented by the matrix X obtained from the corresponding matrix X for the general design in Section 3 above by adding r rows 1, 1, $\cdots$ 1 to it. The matrix $X'X$ for these designs then comes out as

$$X'X = \begin{pmatrix} N & r & r & \cdots & r \\ r & N & r & \cdots & r \\ r & r & N & \cdots & r \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r & r & r & \cdots & N \end{pmatrix} \tag{19}$$

which is of order $p \times p$ for zero bias, or of order $(p + 1) \times (p + 1)$ for non-zero bias. The variance of each unknown determined by this experiment is

$$\frac{\sigma^2}{N - \dfrac{(p-1)r^2}{N + (p-2)r}} \qquad \text{for zero bias,} \tag{20}$$

or

$$\frac{\sigma^2}{N - \dfrac{pr^2}{N + (p-1)r}} \qquad \text{if there is bias.} \tag{21}$$

Hence the efficiency of this design is

$$1 - \frac{(p-1)r^2}{N[N + (p-2)r]} \qquad \text{for zero bias,} \tag{22}$$

or

$$1 - \frac{pr^2}{N[N + (p-1)r]} \qquad \text{if there is bias.} \tag{23}$$

The loss of efficiency as a result of adopting this design is, therefore, $\dfrac{(p-1)r^2}{N[N + (p-2)r]}$ for zero bias, or $\dfrac{pr^2}{N[N + (p-1)r]}$ if there is bias.

7. Second design for $N = 2^m + r$, $p \leq 2^m$ (for zero bias) or $p \leq 2^m - 1$ (for non-zero bias). Another design for these values of N and p is that represented by the matrix X obtained from the corresponding matrix X for the general design in Section 3 above by adding to it r rows $1, 0, 0, \cdots 0$. The matrix $X'X$ for this design is then given by

$$X'X = \begin{pmatrix} N & 0 & 0 & \cdots & 0 \\ 0 & N-r & 0 & \cdots & 0 \\ 0 & 0 & N-r & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & N-r \end{pmatrix}, \tag{24}$$

which is of order $p \times p$ if there is zero bias, or of order $(p + 1) \times (p + 1)$ if there is bias. Here also the estimates of all the unknowns are mutually orthogonal. The efficiency of the design comes out to be

$$\frac{(N-r)p}{Np - r} \qquad \text{if there is zero bias,} \tag{25}$$

or

$$\frac{N-r}{N} \qquad \text{if there is bias.} \tag{26}$$

By comparing the efficiency of the first design of this type given in (22) and (23) with that of the present design given in (25) and (26) respectively, it would appear that in case of zero bias, the efficiency of the first design is higher than that of the second design for $p > 1$, but equal for $p = 1$; and in case of non-zero bias, the efficiency of the first design is always higher than that of the second.

8. Comprehensive general design when N is at our choice. When N is at our choice, we can always obtain a completely orthogonalized design by taking N equal to a sufficiently large power of 2. For $p = 2^m$, m being any positive integer, a completely orthogonalized design for $N = 2^m$, when there is zero bias, has been given in Section 3 above. If, however, there is a bias, a completely orthogonalized design can be constructed for $N = 2^{m+1}$. When $p = 2^m + u$, where u is a positive integer $< 2^m$, a completely orthogonalized design is available for $N = 2^{m+1}$, whether the bias is zero or not.

For $N = 2^{m+1}$, this is the most efficient design, with 100 per cent efficiency, but as N is given higher powers of 2 than 2^{m+1}, the variance of the estimate of each unknown decreases. When $N = 2^l$, where $l > m + 1$, the variance of each unknown is $\frac{1}{2^{l-m-1}}$ of that for $N = 2^{m+1}$.

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