# ON THE USE OF THE SAMPLE RANGE IN AN ANALOGUE OF STUDENT'S $t$-TEST

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Let $x_1, \cdots, x_N$ represent independent observations on a variate $x$ which is normally distributed with mean $\mu$ and variance $\sigma^2$. Assuming no prior information about the value of either parameter, let $H_0$ be the hypothesis that $\mu$ is equal to or less than a specified quantity $\mu_0$. The classical test of this asymmetrical form of "Student's" hypothesis [1] is based upon the statistic

$$t = \sqrt{N}(\bar{x} - \mu_0) \Big/ \sqrt{\frac{\Sigma(x - \bar{x})^2}{N - 1}},$$

the region of rejection being defined by the relation $t > t_\epsilon$.

For certain applications of a routine nature, however, such as production line inspection, the usefulness of this test is rather seriously impaired by the arithmetical work involved in the computation of $t$. For this reason Dodge [2] and Knudsen [3] among others have proposed tests of $H_0$ based on a statistic of the form

$$G = \frac{\bar{x} - \mu_0}{w}$$

where $w$ is the sample range. It is the object of this note to show how the probability distribution of $G$ can be obtained with the aid of the distribution law of $w$ tabulated by Pearson and Hartley [4], and to present some numerical results which indicate that the power of the resulting test is the same for all practical purposes as that of "Student's" $t$-test for sample sizes $N \leq 10$.

The calculation of the percent points of the $G$ distribution is greatly facilitated by the following result, which does not appear to be generally known:

Lemma: *If $\bar{x}$ and $w$ represent respectively the average and the range of a sample of $N$ independent observations on a normally distributed variate $x$, then $\bar{x}$ and $w$ are statistically independent.*

Proof: No generality is lost by putting $\mu = 0$, $\sigma^2 = 1$. The joint characteristic function of $\bar{x}$ and the $\frac{1}{2}N(N - 1)$ differences $x_j - x_k$, $(j < k)$, is then

$$\varphi(t, t_{jk}) = (2\pi)^{-(N/2)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_j x_j^2 + i\frac{t}{N}\sum_j x_j + i\sum_{j,k} t_{jk}(x_j - x_k)} dx_1 \cdots dx_N$$

where the summation runs from 1 to $N$ on each index with the understanding that $t_{jk} \equiv 0$ for $j \geq k$. The usual process of completing the square in the exponent then yields

$$\varphi(t, t_{jk}) = e^{-\frac{1}{2}\sum_j \left[\frac{t}{N} + \sum_k (t_{jk} - t_{kj})\right]^2} \cdot (2\pi)^{-(N/2)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_j \left\{x_j - i\left[\frac{t}{N} + \sum_k (t_{jk} - t_{kj})\right]\right\}^2} dx_1 \cdots dx_N.$$

Since

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\theta)^2} dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx,$$

this reduces to

$$\varphi(t, t_{jk}) = e^{-\frac{1}{2}\sum_i \left[\frac{t}{N}+\sum_k (t_{jk}-t_{kj})\right]^2},$$

which readily factors into

$$\varphi_1(t)\cdot\varphi_2(t_{jk}) = e^{-(t^2/2N)}\cdot e^{-\frac{1}{2}\sum_i\left[\sum_k (t_{jk}-t_{kj})\right]^2}.$$

Hence the differences $x_j - x_k$ are jointly independent of $\bar{x}$; and since the range $w$ is a Borel measurable function of these differences (i.e., $w = \max | x_j - x_k |$) it follows that $\bar{x}$ and $w$ are independently distributed.

The foregoing lemma is in fact capable of further generalization as follows: *Let $g(x_1, \cdots, x_N)$ be a function which, like the range, has the property that $g(x_1 + a, \cdots, x_N + a) \equiv g(x_1, \cdots, x_N)$. The characteristic function of $\bar{x}$ and $g$ can then be written in the form*

$$\varphi(t,\lambda) = e^{-(t^2/2N)}\cdot(2\pi)^{-(N/2)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum(x-i(t/N))^2+i\lambda g(x)}\,dx_1\cdots dx_N = \varphi_1(t)\cdot\psi(t,\lambda).$$

Now if the second factor $\psi$ is analytic in $t$, it must be a constant as far as variation with $t$ is concerned; for by putting $t = iNa$ ($a$ real) we have

$$\begin{aligned}\psi(iNa,\lambda) &= (2\pi)^{-(N/2)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum(x+a)^2+i\lambda g(x)}\,dx_1\cdots dx_N\\ &= (2\pi)^{-(N/2)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum(x+a)^2+i\lambda g(x+a)}\,dx_1\cdots dx_N\\ &= (2\pi)^{-(N/2)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum z^2+i\lambda g(z)}\,dz_1\cdots dz_N = \varphi_2(\lambda).\end{aligned}$$

Therefore $\psi(t, \lambda)$, being constant in $t$ along the axis of imaginaries, must be free of $t$ throughout the complex plane. The joint characteristic function of $\bar{x}$ and $g$ is thus equal to the product of their respective characteristic functions, so that the two variates are independently distributed. In particular this result shows that in the normal case each of the moments about the sample mean is distributed independently of $\bar{x}$.

Returning now to the distribution of $G$, we see that for $G_\epsilon > 0$

$$\begin{aligned}P\left\{\frac{\bar{x}-\mu}{w} > G_\epsilon\right\} &= P\left\{\frac{\sqrt{N}(\bar{x}-\mu)/\sigma}{\sqrt{N}G_\epsilon} > w/\sigma\right\}\\ &= \int_{z=0}^{\infty}\int_{w=0}^{z/\sqrt{N}G_\epsilon} f(z)h(w)\,dw\,dz\\ &= \int_0^{\infty} f(z)P(z/\sqrt{N}G_\epsilon)\,dz\end{aligned}$$

where $f(z)$ is the normal probability function for $\mu = 0$, $\sigma^2 = 1$, and $P(u)$ is the value [4] of the probability that the range of a sample of $N$ observations will be less than $u$ standard units. For selected values of $N$ Table I gives the value $G_{.05}$ such that

$$P_N\{(\bar{x} - \mu_0)/w > G_{.05} \mid \mu = \mu_0\} = .05.$$

TABLE I

*Upper 5% points for distribution of G*

| N | $G_{.05}$ |
|---|---|
| 3 | .88 |
| 5 | .39 |
| 7 | .26 |
| 10 | .19 |

These values were calculated by Simpson's rule and checked by Weddle's rule.

To evaluate the probability that $G$ will exceed $G_\epsilon$ when $\mu \neq \mu_0$ we may write, following Johnson and Welch [5]

$$\frac{\bar{x} - \mu_0}{w} = \frac{\sqrt{N}(\bar{x} - \mu)/\sigma + \sqrt{N}(\mu - \mu_0)/\sigma}{\sqrt{N}w/\sigma} = \frac{z + a}{\sqrt{N}w/\sigma}.$$

The required probability is then given by the integral

$$\int_{z=-a}^{\infty} f(z)P\left(\frac{z + a}{\sqrt{N}G_\epsilon}\right) dz, \quad a = \sqrt{N}(\mu - \mu_0)/\sigma.$$

Table II is a comparison of the probability that $G$ will exceed $G_{.05}$ with the corresponding probability that "Student's" $t$ will exceed $t_{.05}$ for various values of $(\mu - \mu_0)/\sigma$, the case $N = 3$ being chosen because the non-central $t$ distribution is formally integrable in this case.

TABLE II

*Probability of rejection for G and for t, (N = 3)*

| $(\mu - \mu_0)/\sigma$ | $P\{G > .88\}$ | $P\{t > 2.92\}$ |
|---|---|---|
| .00 | .050 | .050 |
| .50 | .151 | .151 |
| .75 | .229 | .230 |
| 1.00 | .322 | .322 |

Similarly for $N = 10$ it was found that when $\mu - \mu_0 = .383\sigma$ (i.e., when $a = 1.21$) the probability that $G$ will exceed $G_{.05}$ is .296; the corresponding probability for $t$ is given by Neyman and Tokarska [1] as .30.

Pending the construction of more adequate tables of the percent points of the $G$ distribution, it seems worthy of note that for $N \leq 10$ the values of $G_{.05}$ can be estimated quite accurately by multiplying the corresponding upper percent point $t_{.05}$ by the factor

$$k_N = \frac{E\left[\sqrt{\frac{\Sigma(x - \bar{x})^2}{N - 1}}\right]}{\sqrt{N}E[w]}$$

where $E[w]$ is obtainable from Tippett's table of the mean range [6]. Estimated values of $G_{.05}$ for sample sizes from 3 to 10 are listed for convenience in Table III. The approximate values of $G_{.05}$ proposed by Knudsen [3] were calculated in essentially this fashion, using however the square root of the expected value of $\Sigma(x - \bar{x})^2$ instead of the expected value of $\sqrt{\Sigma(x - \bar{x})^2}$, and employing percent points of the $t$ distribution determined by the relation $P\{|t| > t^*_{.05}\} = .05$ instead of $P\{t > t_{.05}\} = .05$. Thus though the agreement between the values listed in Table III and the corresponding computed values shown in Table I is extremely good, the discrepancy between these values and those given by Knudsen is rather large. Any error committed by using Knudsen's table will,

TABLE III

*Estimated upper 5% points for distribution of G*

| N | $G_{.05}$ |
|---|---|
| 3 | .882 |
| 4 | .526 |
| 5 | .385 |
| 6 | .309 |
| 7 | .260 |
| 8 | .227 |
| 9 | .202 |
| 10 | .183 |

however, be on the conservative side, in the sense that the probability of unjustly rejecting $H_0$ will have somewhat less than half the value indicated in that table.

REFERENCES

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