lute odd moments of all orders are uniformly bounded, a bound for the absolute moments of order $2k - 1$ being one greater than the absolute moment of this order of H_k. This in turn insures that the odd moments of $H^*(x)$ exist and that they have the desired values. By adding a jump of $1 - H^*(\infty)$ at the origin we obtain $H(x)$, a c.d.f. with the given odd moments.

The main statement of this note is an immediate consequence of the lemma. Let the kth odd moment of $F(x)$ be M_{2k-1}, which we assume to be finite, and let the sequence $\{m_{2k-1}\}$ be defined by the relationships:

$$\mu_{2k-1} = (1 - \epsilon)M_{2k-1} + \epsilon m_{2k-1}, \qquad (k = 1,2, \cdots).$$

Let $H(x)$ have the m's as odd moments. The c.d.f. $F^*(x)$ defined by

$$F^*(x) = (1 - \epsilon)F(x) + \epsilon H(x)$$

clearly has the properties stated above, and our statement is proved. If the moments of $F(x)$ are not all finite, the proof will need only minor modifications.

If one asks in addition that F^* have a finite range, F^* will, in general, not exist. If, for example, the range of F is finite and its odd moments are zero, then F must be symmetric about the origin, for F^* defined by $dF^*(x) = dF(-x)$ would have the same moments as F. But a c.d.f. with finite range is determined by its moments; hence $F(x) = F^*(x)$.

SOME ORDER STATISTIC DISTRIBUTIONS FOR SAMPLES OF SIZE FOUR

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1. Summary. Let x_1, x_2, x_3, x_4 represent the values of a sample of size four drawn from a normal population. There is no loss of generality in assuming that the distribution function of this population has zero mean and unit variance. Denote it by $N(0, 1)$. Let $x_{(i)}$ be the ith largest of x_1, x_2, x_3, x_4. The purpose of this note is to determine the joint distribution of

$x_{(4)} + x_{(3)} - x_{(2)} - x_{(1)}$, $x_{(4)} - x_{(3)} + x_{(2)} - x_{(1)}$, and $x_{(4)} - x_{(3)} - x_{(2)} + x_{(1)}$, and derive from this joint distribution the joint distributions of these statistics taken in pairs, also the distribution of each statistic itself.

2. Analysis. Consider the joint distribution of

$$r_1 = \tfrac{1}{2}(x_4 + x_3 - x_2 - x_1)$$

$$r_2 = \tfrac{1}{2}(x_4 - x_3 + x_2 - x_1)$$

$$r_3 = \tfrac{1}{2}(x_4 - x_3 - x_2 + x_1).$$

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Evidently,

$$E(r_i) = 0, \quad (i = 1, 2, 3). \quad E(r_i r_j) = 0, \quad (i \neq j). \quad E(r_i^2) = 1.$$

Hence the r_i are independently distributed according to $N(0, 1)$.

Let v_j be the jth largest of $|r_1|$, $|r_2|$, $|r_3|$. Then by first finding the joint distribution of $|r_1|$, $|r_2|$, $|r_3|$ and then applying the distribution for order statistics [1], it is easily seen that the joint distribution element of v_1, v_2, v_3 is

$$48f(v_1)f(v_2)f(v_3)dv_1dv_2dv_3,$$

where

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}, \qquad 0 \leq v_1 \leq v_2 \leq v_3.$$

Examination shows, however, that

$$\begin{aligned} v_3 &= \tfrac{1}{2}(x_{(4)} + x_{(3)} - x_{(2)} - x_{(1)}) \\ v_2 &= \tfrac{1}{2}(x_{(4)} - x_{(3)} + x_{(2)} - x_{(1)}) \\ v_1 &= \tfrac{1}{2} \,|\, x_{(4)} - x_{(3)} - x_{(2)} + x_{(1)} \,|. \end{aligned}$$

Let

$$\begin{aligned} m_3 &= x_{(4)} + x_{(3)} - x_{(2)} - x_{(1)} \\ m_2 &= x_{(4)} - x_{(3)} + x_{(2)} - x_{(1)} \\ m_1 &= x_{(4)} - x_{(3)} - x_{(2)} + x_{(1)}. \end{aligned}$$

Then the joint distribution element of $|m_1|$, m_2 and m_3 is

$$6f(\tfrac{1}{2}\,|\,m_1\,|)f(\tfrac{1}{2}m_2)f(\tfrac{1}{2}m_3)d\,|\,m_1\,|\,dm_2dm_3.$$

Since the function f is symmetrical about the origin, it follows immediately that the joint distribution element of m_1, m_2 and m_3 is

$$3f(\tfrac{1}{2}m_1)f(\tfrac{1}{2}m_2)f(\tfrac{1}{2}m_3)dm_1dm_2dm_3,$$

where $|\,m_1\,| \leq m_2 \leq m_3$.

3. Derived results. By taking marginal distributions it is found that the joint distribution elements of m_1, m_2 and m_3 taken in pairs are

$$g_1(m_1, m_2)dm_1\,dm_2 = 3\left(\int_{m_2}^{\infty} f(\tfrac{1}{2}y)dy\right) f(\tfrac{1}{2}m_1)f(\tfrac{1}{2}m_2)dm_1\,dm_2.$$

$$g_2(m_1, m_3)dm_1\,dm_3 = 3\left(\int_{|m_1|}^{m_3} f(\tfrac{1}{2}y)dy\right) f(\tfrac{1}{2}m_1)f(\tfrac{1}{2}m_3)dm_1\,dm_3.$$

$$g_3(m_2, m_3)dm_2\,dm_3 = 6\left(\int_{0}^{m_2} f(\tfrac{1}{2}y)dy\right) f(\tfrac{1}{2}m_2)f(\tfrac{1}{2}m_3)dm_2\,dm_3.$$

The distribution elements of m_1, m_2 and m_3 are seen to be

$$g_1(m_1)dm_1 = \frac{3}{2}\left(\int_{|m_1|}^{\infty} f(\tfrac{1}{2}y)dy\right)^2 f(\tfrac{1}{2}m_1)dm_1.$$

$$g_2(m_2)dm_2 = 6\left(\int_0^{m_2} f(\tfrac{1}{2}y)dy\right)\left(\int_{m_2}^{\infty} f(\tfrac{1}{2}y)dy\right)f(\tfrac{1}{2}m_2)dm_2.$$

$$g_3(m_3)dm_3 = 3\left(\int_0^{m_3} f(\tfrac{1}{2}y)dy\right)^2 f(\tfrac{1}{2}m_3)dm_3.$$

It is to be noted that if $a > 0$,

$$Pr(0 < m_1 < a) = Pr(-a < m_1 < 0) = \tfrac{1}{2} - 4\left(\int_{a/2}^{\infty} f(y)dy\right)^3,$$

$$Pr(0 < m_2 < a) = 12\left(\int_0^{a/2} f(y)dy\right)^2 - 16\left(\int_0^{a/2} f(y)dy\right)^3,$$

$$Pr(0 < m_3 < a) = 8\left(\int_0^{a/2} f(y)dy\right)^3,$$

so that the probability that any of m_1, m_2, m_3 lie between two given numbers is expressed explicitly and can be calculated with the aid of standard tables for the normal distribution.

4. Generalization of method. The method used to obtain the joint distribution of the order statistics m_1, m_2 and m_3 was to take all possible combinations of 4 variables with two plus and two minus signs (except for factor of -1) and show that these combinations behave as normally distributed independent variables. The question arises as to whether this method of finding order statistic distributions would apply in general to $2n$ variables with n plus and n minus signs. It is easily proved that this will occur only when $n = 2$.

REFERENCES

[1] S. S. Wilks, *Mathematical Statistics*, Princeton Univ. Press, 1943, p. 90.