TOLERANCE LIMITS FOR A NORMAL DISTRIBUTION[1]

By A. Wald and J. Wolfowitz

Columbia University and *University of North Carolina*

Summary. The problem of constructing tolerance limits for a normal universe is considered. The tolerance limits are required to be such that the probability is equal to a preassigned value β that the tolerance limits include at least a given proportion γ of the population. A good approximation to such tolerance limits can be obtained as follows: Let $\bar{x}$ denote the sample mean and s^2 the sample estimate of the variance. Then the approximate tolerance limits are given by

$$\bar{x} - \sqrt{\frac{n}{\chi^2_{n,\beta}}}\, rs \quad \text{and} \quad \bar{x} + \sqrt{\frac{n}{\chi^2_{n,\beta}}}\, rs$$

where n is one less than the number N of observations, $\chi^2_{n,\beta}$ denotes the number for which the probability that χ^2 with n degrees of freedom will exceed this number is β, and r is the root of the equation

$$\frac{1}{\sqrt{2\pi}} \int_{1/\sqrt{N}-r}^{1/\sqrt{N}+r} e^{-t^2/2}\, dt = \gamma.$$

The number $\chi^2_{n,\beta}$ can be obtained from a table of the χ^2 distribution and r can be determined with the help of a table of the normal distribution.

1. Introduction. The problem of setting tolerance limits for a distribution on the basis of an observed sample was discussed by S. S. Wilks [1], [2] and by one of the present authors [3], [4]. For a univariate distribution the problem may be formulated briefly as follows: Let x be the chance variable under consideration and let $x_1, \cdots, x_N$ be a sample of N independent observations on x. Two functions, L_1 and L_2, of the sample are to be constructed such that the probability that the limits L_1 and L_2 will include at least a given proportion γ of the population is equal to a preassigned value β. The limits L_1 and L_2 are called tolerance limits.

The following two cases have been treated in the literature: (1) Nothing is known about the distribution of x, except perhaps that it is continuous, or that it admits a continuous probability density function. (2) The functional form of the distribution of x is known and only the values of a finite number of parameters involved in the distribution of x are unknown. We shall refer to (1) as the non-

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parametric case and to (2) as the parametric case. An exact solution of the problem for univariate distributions in the non-parametric case has been given by S. S. Wilks [1]. His results have been extended to multivariate distributions by one of the present authors [3]. An asymptotic solution of the problem in the parametric case, which may be used for large samples, was given in [4].[2]

In the present paper we shall deal with the problem of setting tolerance limits for a normal distribution with unknown mean and variance. Approximation formulas are obtained which differ from the exact values by a magnitude of the order $1/N^2$. They give much closer approximations to the exact values than those which can be obtained by applying the general asymptotic results in [4] to the normal distribution. In addition, the approximation formulas in the present paper have the advantage of considerable simplicity and can easily be computed with the help of tables of the normal and χ^2 distributions. To estimate the closeness of the approximation of the formulas given in this paper, a method of computing upper and lower limits for the exact values has been derived. Computations show that the approximation is good even for small values of N. A few numerical examples are given in section 7.

2. Precise formulation of the problem and notation. Let $x_1, \cdots, x_N$ be N independent observations from a normal population with mean μ and variance σ^2, both unknown. We shall denote by $\bar{x}$ the arithmetic mean of the observations and by s^2 the sample estimate of the population variance σ^2, i.e.,

$$\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} \tag{2.1}$$

and

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n}, \quad \text{where } n = N - 1. \tag{2.2}$$

For any positive λ we shall denote by $A(\bar{x}, s, \lambda)$, or more briefly by A, the proportion of the normal universe included between the limits $\bar{x} - \lambda s$ and $\bar{x} + \lambda s$, i.e.,

$$A = A(\bar{x}, s, \lambda) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\bar{x}-\lambda s}^{\bar{x}+\lambda s} e^{-(1/2\sigma^2)(t-\mu)^2}\, dt. \tag{2.3}$$

A is a chance variable, since the limits of integration are chance variables. In this paper we shall deal with the problem of determining the value of λ so that the probability that A exceeds a preassigned value γ is equal to a preassigned value β. The desired tolerance limits will then be given by $\bar{x} - \lambda s$ and $\bar{x} + \lambda s$, respectively. In practice, the values β and γ will usually be chosen near unity, frequently $\geq .95$.

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[2] Although the results obtained in the non-parametric case could be applied to the parametric case as well, it would not be satisfactory to do so, since for the parametric case methods having greater efficiency can be devised by taking into account the available information regarding the functional form of the distribution.

It can be verified that the distribution of A does not depend on the unknown parameters μ and σ. Thus we can assume without loss of generality that $\mu = 0$ and $\sigma = 1$.

For any given positive value λ we shall denote by $P(\gamma,\lambda)$ the probability that $A > \gamma$. For a given value $\bar{x}$ we shall denote by $P(\gamma,\lambda \mid \bar{x})$ the conditional probability that $A > \gamma$ under the condition that the sample mean has a given value $\bar{x}$. It is clear that $P(\gamma,\lambda)$ is equal to the expected value of $P(\gamma,\lambda \mid \bar{x})$, i.e.,

$$P(\gamma,\lambda) = \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} P(\gamma, \lambda \mid \bar{x})\, e^{-\frac{1}{2}N\bar{x}^2}\, d\bar{x}\,. \tag{2.4}$$

3. Method of computing $P(\gamma,\lambda \mid \bar{x})$ for any given values γ,λ and $\bar{x}$. Since $A = A(\bar{x},s,\lambda)$ is a strictly increasing function of s, the equation in s

$$A(\bar{x},s,\lambda) = \gamma \tag{3.1}$$

has exactly one root in s. Denote this root by

$$s = r(\bar{x},\gamma,\lambda). \tag{3.2}$$

Thus, $r(\bar{x},\gamma,\lambda)$ is that value for which

$$\frac{1}{\sqrt{2\pi}} \int_{\bar{x}-\lambda r(\bar{x},\gamma,\lambda)}^{\bar{x}+\lambda r(\bar{x},\gamma,\lambda)} e^{-t^2/2}\, dt = \gamma. \tag{3.3}$$

It is clear that $\lambda r(\bar{x},\gamma,\lambda)$ does not depend on λ. We shall write

$$\lambda r(\bar{x},\gamma,\lambda) = r(\bar{x},\gamma). \tag{3.4}$$

Obviously $r(\bar{x},\gamma)$ is that value for which

$$\frac{1}{\sqrt{2\pi}} \int_{\bar{x}-r(\bar{x},\gamma)}^{\bar{x}+r(\bar{x},\gamma)} e^{-t^2/2}\, dt = \gamma. \tag{3.5}$$

For given values of $\bar{x}$ and γ the value $r(\bar{x},\gamma)$ can be obtained from a table of the normal distribution.

Since $A(\bar{x},s,\lambda)$ is a strictly increasing function of s, the inequality $A(\bar{x},s,\lambda) > \gamma$ is equivalent to the inequality $s > r(\bar{x},\gamma,\lambda) = r(\bar{x},\gamma)/\lambda$. Hence, since $\bar{x}$ and s are independently distributed, we have

$$P(\gamma,\lambda \mid \bar{x}) = P(s > r(\bar{x},\gamma)/\lambda) \tag{3.6}$$

where $P(s > c)$ denotes the probability that $s > c$ for any constant c. In general, for any relation R we shall denote by $P(R)$ the probability that R holds.

Since ns^2 has the χ^2 distribution with $n = N - 1$ degrees of freedom, we have

$$P\left(s > \frac{r(\bar{x}, \gamma)}{\lambda}\right) = P\left(\chi_n^2 > \frac{nr^2(\bar{x}, \gamma)}{\lambda^2}\right) \tag{3.7}$$

where χ_n^2 stands for a random variable which has the χ^2 distribution with n degrees of freedom. The probability on the right-hand side of (3.7) can be obtained from a table of the χ^2 distribution.

Hence, we see that the computation of $P(\gamma,\lambda \mid \bar{x})$ for given values γ,λ and $\bar{x}$ can be carried out in two simple steps. First we determine the value of $r(\bar{x},\gamma)$ from a table of the normal distribution and then read the value of

$$P\left(\chi_n^2 > \frac{nr^2(\bar{x},\gamma)}{\lambda^2}\right)$$

from a table of the χ^2 distribution.

4. Proof that the difference $P\left(\gamma,\lambda \,\middle|\, \frac{1}{\sqrt{N}}\right) - P(\gamma,\lambda)$ is of the order $1/N^2$. It is clear that $P(\gamma,\lambda \mid \bar{x})$ is an even function of $\bar{x}$. Hence, in the expansion of $P(\gamma,\lambda \mid \bar{x})$ in a power series in $\bar{x}$, only even powers will occur. Terminating the Taylor expansion (in section 8 we prove its validity) at the fourth term, we have

$$(4.1)\quad P(\gamma,\lambda \mid \bar{x}) = P(\gamma,\lambda \mid 0) + \frac{\bar{x}^2}{2}\frac{\partial^2 P(\gamma,\lambda \mid \bar{x})}{\partial \bar{x}^2}\bigg|_{\bar{x}=0} + \frac{\bar{x}^4}{4!}\frac{\partial^4 P(\gamma,\lambda \mid \bar{x})}{\partial \bar{x}^4}\bigg|_{\bar{x}=\xi}$$

where $0 \le \xi \le \bar{x}$.

The expected value of $P(\gamma,\lambda \mid \bar{x})$ (considering $\bar{x}$ as a random variable) is equal to $P(\gamma,\lambda)$. Since the expected value of $\bar{x}^2$ is $1/N$ and the expected value of

$$\frac{\bar{x}^4}{4!}\frac{\partial^4 P}{\partial \bar{x}^4}\bigg|_{\bar{x}=\xi}$$

is of the order $1/N^2$ (this is proved in section 9), we obtain from (4.1)

$$(4.2)\qquad P(\gamma,\lambda) = P(\gamma,\lambda \mid 0) + \frac{1}{2N}\frac{\partial^2 P}{\partial \bar{x}^2}\bigg|_{\bar{x}=0} + 0\left(\frac{1}{N^2}\right).$$

On the other hand, substituting $1/\sqrt{N}$ for $\bar{x}$ in (4.1) we obtain

$$(4.3)\qquad P\left(\gamma,\lambda \,\middle|\, \frac{1}{\sqrt{N}}\right) = P(\gamma,\lambda \mid 0) + \frac{1}{2N}\frac{\partial^2 P}{\partial \bar{x}^2}\bigg|_{\bar{x}=0} + \frac{1}{4!N^2}\frac{\partial^4 P}{\partial \bar{x}^4}\bigg|_{\bar{x}=\xi'}$$

where $0 \le \xi' \le 1/\sqrt{N}$. Hence, since the second term of the right member of (4.3) is of the order $1/N^2$,

$$(4.4)\qquad P\left(\gamma,\lambda \,\middle|\, \frac{1}{\sqrt{N}}\right) = P(\gamma,\lambda \mid 0) + \frac{1}{2N}\frac{\partial^2 P}{\partial \bar{x}^2}\bigg|_{\bar{x}=0} + 0\left(\frac{1}{N^2}\right).$$

From (4.2) and (4.4) it follows that

$$(4.5)\qquad P(\gamma,\lambda) - P\left(\gamma,\lambda \,\middle|\, \frac{1}{\sqrt{N}}\right) = 0\left(\frac{1}{N^2}\right).$$

Thus, this difference approaches zero rapidly as $N \to \infty$.

5. Computation of the value λ for which $P\left(\gamma,\lambda \,\middle|\, \frac{1}{\sqrt{N}}\right)$ takes a preassigned value β. Denote by $\chi^2_{n,\beta}$ that value for which $P(\chi_n^2 > \chi^2_{n,\beta}) = \beta$. This value can be obtained from a table of the χ^2 distribution. From (3.6) and (3.7) it follows that the required value λ^* of λ is given by the root of the equation

$$\frac{n}{\lambda^2} r^2\left(\frac{1}{\sqrt{N}}, \gamma\right) = \chi^2_{n,\beta}. \tag{5.1}$$

Thus, the desired value of λ^* is given by

$$\lambda^* = \sqrt{\frac{n}{\chi^2_{n,\beta}}}\, r\left(\frac{1}{\sqrt{N}}, \gamma\right). \tag{5.2}$$

The value $r\left(\frac{1}{\sqrt{N}}, \gamma\right)$ is defined by (3.5) and can be obtained from a table of the normal distribution.[3]

6. Lower and upper limits for $P(\gamma,\lambda)$. As mentioned in section 2, $P(\gamma,\lambda)$ is equal to the expected value of $P(\gamma,\lambda \mid \bar{x})$. Thus,

$$P(\gamma,\lambda) = \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} P(\gamma,\lambda \mid \bar{x}) e^{-\frac{1}{2}N\bar{x}^2}\, d\bar{x}. \tag{6.1}$$

To obtain upper and lower limits for $P(\gamma,\lambda)$, we shall construct upper and lower limits for the integral on the right-hand side of (6.1). It can easily be seen that $P(\gamma,\lambda \mid \bar{x})$ is a strictly decreasing function of $\bar{x}^2$. Hence, to obtain lower and upper limits for the integral in the right member of (6.1) we can proceed as follows: Choose a positive constant d and a positive integer k. Denote by a_i the probability that $id \leq \bar{x} \leq (i + 1)d$, $(i = 0, 1, \cdots, k-1)$, and let a_k be the probability that $\bar{x} > kd$. Then $2\sum_{i=0}^{k} a_i P(\gamma,\lambda \mid id)$ is an upper bound, and $2\sum_{i=1}^{k} a_{i-1} P(\gamma,\lambda \mid id)$ is a lower bound of the integral in question. Thus

$$P(\gamma,\lambda) \geq 2 \sum_{i=1}^{k} a_{i-1} P(\gamma,\lambda \mid id) \tag{6.2}$$

and

$$P(\gamma, \lambda) \leq 2 \sum_{i=0}^{k} a_i P(\gamma,\lambda \mid id). \tag{6.3}$$

The two limits can be brought arbitrarily close to each other by choosing d sufficiently small and k sufficiently large. A method of computing $P(\gamma,\lambda \mid \bar{x})$ for any given value $\bar{x}$ has been described in section 3 and the quantities a_i can be obtained from a table of the normal distribution. The amount of computational work, however, increases rapidly with increasing k.

[3] The Statistical Research Group computed, under the supervision of Albert H. Bowker, a table of tolerance limit factors λ (see formula 5.2) for $\beta = .75, .90, .95, .99$; $\gamma = .75, .90, .95, .99, .999$; $N = 2\ (1)\ 102\ (2)\ 180\ (5)\ 300\ (10)\ 400\ (25)\ 750\ (50)\ 1000$. Mr. Bowker also developed an asymptotic formula for λ (published elsewhere in this issue of the *Annals*) which, when $\beta \leq .99$, $\gamma \leq .999$, and $N \geq 160$, agrees with (5.2) to within 1 unit in the third significant figure. The Applied Mathematics Panel plans to publish the table and a brief explanation of tolerance limits in the volume entitled *Techniques of Statistical Analysis* described in the footnote on page 217.

7. Approximate determination of the tolerance limits. The exact tolerance limits are given by $\bar{x} - \lambda s$ and $\bar{x} + \lambda s$ where λ is the root of the equation in λ

$$P(\gamma,\lambda) = \beta. \tag{7.1}$$

This equation has exactly one root in λ, since $P(\gamma,\lambda)$ is a strictly increasing function of λ. Denote this root by $\lambda = \lambda(\beta,\gamma)$. Thus, the exact tolerance limits are given by $\bar{x} - \lambda(\beta,\gamma)s$ and $\bar{x} + \lambda(\beta,\gamma)s$.

We have seen in section 4 that $P\left(\gamma,\lambda \mid \frac{1}{\sqrt{N}}\right)$ closely approximates $P(\gamma,\lambda)$, the difference being of the order $1/N^2$. Thus, a close approximation to $\lambda(\beta,\gamma)$ can be obtained by solving the equation in λ,

$$P\left(\gamma, \lambda \mid \frac{1}{\sqrt{N}}\right) = \beta. \tag{7.2}$$

This equation has again exactly one root in λ, since $P\left(\gamma,\lambda \mid \frac{1}{\sqrt{N}}\right)$ is a strictly increasing function of λ. Denote the root of equation (7.2) by $\lambda = \lambda^*(\beta,\gamma)$. Thus approximate tolerance limits are given by $\bar{x}-\lambda^*(\beta,\gamma)s$ and $\bar{x}+\lambda^*(\beta,\gamma)s$. In section 5 it has been shown that

$$\lambda^*(\beta,\gamma) = \sqrt{\frac{n}{\chi^2_{n,\beta}}}\, r \tag{7.3}$$

where $n = N-1$, $\chi^2_{n,\beta}$ is that number for which the probability that χ^2 with n degrees of freedom exceeds this number is β, and r is the root of the equation

$$\frac{1}{\sqrt{2\pi}} \int_{1/\sqrt{N}-r}^{1/\sqrt{N}+r} e^{-t^2/2}\, dt = \gamma\,. \tag{7.4}$$

The number $\chi^2_{n,\beta}$ can be obtained from a table of the χ^2 distribution and r can be determined from a table of the normal distribution.

Since $\lambda^*(\beta,\gamma)$ is only an approximation to $\lambda(\beta,\gamma)$, $P[\gamma,\lambda^*(\beta,\gamma)]$ will differ slightly from β. To judge the goodness of the approximation of $\lambda^*(\beta,\gamma)$ to the exact value $\lambda(\beta,\gamma)$, it is desirable to derive upper and lower limits for the difference $P[\gamma,\lambda^*(\beta,\gamma)] - \beta$. Such limits can be obtained by computing upper and lower limits for $P[\gamma,\lambda^*(\beta,\gamma)]$ using the method described in section 6.

We cite here a few numerical examples to show the goodness of the approximation.

N	γ	β	$\lambda^*(\beta,\gamma)$	Upper limit of $P[\gamma,\lambda^*(\beta,\gamma)]$	Lower limit of $P[\gamma,\lambda^*(\beta,\gamma)]$
2	.95	.95	37.674	.95202	.95077
9	.95	.99	4.550	.98989	.98908
25	.95	.95	2.631	.95161	.94393
25	.95	.99	2.972	.99024	.98813

8. Validity of the Taylor expansion of $P(\gamma,\lambda \mid \bar{x})$. We shall show that $P(\gamma,\lambda \mid \bar{x})$ has derivatives of all orders at every point $\bar{x}$, γ and λ being fixed. This is sufficient to validate the Taylor expansion used in section 4.

For typographical convenience write

$$r(\bar{x},\gamma) = R.$$

We have

$$\frac{1}{\sqrt{2\pi}} \int_{\bar{x}-R}^{\bar{x}+R} e^{-\frac{1}{2}t^2}\,dt = \gamma. \tag{8.1}$$

Differentiating (8.1) with respect to $\bar{x}$ we obtain

$$\left(1 + \frac{dR}{d\bar{x}}\right) e^{-\frac{1}{2}(\bar{x}+R)^2} = \left(1 - \frac{dR}{d\bar{x}}\right) e^{-\frac{1}{2}(\bar{x}-R)^2} \tag{8.2}$$

whence

$$\frac{dR}{d\bar{x}} = \tanh \bar{x}R\,. \tag{8.3}$$

Now the analytic function $\tanh z$ of the complex variable z has only purely imaginary singularities. Hence R possesses derivatives of all orders for all real values of $\bar{x}$.

Now

$$P(\gamma,\lambda \mid \bar{x}) = P\left(s > \frac{R}{\lambda}\right) = 1 - k\int_0^R t^{n-1}\,e^{-nt^2/(2\lambda^2)}\,dt$$

where k is a constant. Hence from (8.3)

$$\frac{\partial P}{\partial \bar{x}} = -kR^{n-1}\,e^{-nR^2/(2\lambda^2)} \tanh \bar{x}R\,. \tag{8.4}$$

The right member of (8.4) is a product of functions which are analytic in the entire (complex) R plane by a function which possesses derivatives of all orders for every real $\bar{x}$. Since R possesses a derivative (with respect to $\bar{x}$) for all real $\bar{x}$, it follows that P possesses derivatives of all orders for every real $\bar{x}$.

9. Proof that

$$E\left[\frac{\bar{x}^4}{4!}\frac{\partial^4 P}{\partial \bar{x}^4}\bigg|_{\bar{x}=\xi}\right] = 0\left(\frac{1}{N^2}\right).$$

Since R is a minimum at $\bar{x} = 0$ it follows that $P(\gamma,\lambda \mid \bar{x})$ has a maximum there. Hence, from (4.1), the quantity

$$\bar{x}^2\left(\frac{1}{2}\right)\frac{\partial^2 P}{\partial \bar{x}^2}\bigg|_{\bar{x}=0} + \frac{\bar{x}^4}{4!}\frac{\partial^4 P}{\partial \bar{x}^4}\bigg|_{\bar{x}=\xi}$$

is never positive. Therefore

$$\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi} \leq -\frac{12}{x^2}\left.\frac{\partial^2 P}{\partial \bar{x}^2}\right|_{\bar{x}=0}.$$

Consequently $\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi}$ is bounded above for $|\bar{x}| \geq \delta$, where $\delta > 0$ is arbitrarily small. Since P possesses everywhere derivatives of all orders, the fourth derivative is continuous and hence bounded above for $|\bar{x}| \leq \delta$. From this we obtain that $\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi}$ is bounded above for every real $\bar{x}$.

Since $P(\gamma,\lambda \mid \bar{x})$ is always positive we have, from (4.1), that

$$\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi} \geq -\frac{12\left(2P + \bar{x}^2 \left.\frac{\partial^2 P}{\partial \bar{x}^2}\right|_{\bar{x}=0}\right)}{\bar{x}^4}.$$

For $|\bar{x}|$ greater than a sufficiently large number C, the left member of the above inequality is thus bounded below. For $|\bar{x}| \leq C$ we have that $\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi}$ is bounded below because $\frac{\partial^4 P}{\partial \bar{x}^4}$ is continuous. Hence $\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi}$ is bounded below for every real $\bar{x}$.

Since $\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi}$ is bounded above and below for every real $\bar{x}$, the desired result follows.

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