CONCERNING THE EFFECT OF INTRACLASS CORRELATION ON CERTAIN SIGNIFICANCE TESTS

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1. Summary. In practical applications it is frequently assumed that the values obtained by a sampling process are independently drawn from the same normal population. Then confidence intervals and significance tests which were derived under the assumption of independence are applied using these values. Often the assumption of independence between the values may be at best only approximately valid. For some cases, however, it may be permissible to assume that the correlation between each two values is the same (intraclass correlation). The purpose of this paper is to investigate the effect of this intraclass correlation on the confidence coefficients and significance levels of several well known confidence intervals and significance tests which were derived under the assumption of independence, and to extend these considerations to the case of two sets of values.

In the first part of the paper the relations given in Table I are used to compute tables which show the effect of intraclass correlation on the confidence coefficients and significance levels of the confidence intervals and significance tests listed in Table II. The second part of the paper consists of the proofs of the relations given in Table I.

2. Introduction. Let the n values $x_1, \ldots, x_n$ represent a single value of a normal multivariate population for which each of the n variables has mean μ variance σ^2, and the correlation between each two variables is ρ. These n values will be called a correlated "sample." The values $x_1, \cdots, x_n$ and $y_1, \cdots, y_m$ are said to represent two correlated "samples" if they have a normal multivariate distribution such that the x's have mean μ, variance σ^2, correlation ρ, the y's have mean μ', variance σ'^2, correlation ρ', and the correlation between each x and y is ρ''. This paper shows that several well known quantities which have Student t, χ^2, or Snedecor F distributions when the values form random samples still have these same distributions for correlated "samples" if the quantities are multiplied by suitable constant factors, where it is to be remembered that for normal populations a correlated "sample" is a random sample if and only if $\rho = 0$ and that two correlated "samples" represent two random samples if and only if $\rho = \rho' = \rho'' = 0$. The quantities considered and the corresponding factors are listed in Table I, where $\bar{x} = \sum_1^n x_i/n$ and $\bar{y} = \sum_1^m y_\alpha/m$. Several commonly used confidence intervals and significance tests based on these quantities and derived under the assumption of randomness are considered, and tables are computed which show how the confidence coefficients and significance levels of these confidence intervals and significance tests vary if the values are from correlated "samples" instead of random samples. Table II contains an outline of the confidence intervals and significance tests considered. It is found that these confidence coefficients and significance levels can change noticeably when a correlated "sample" is considered. This is particularly true for the Student t-test. For example, in one case it is found that if the sample size is 32 and the significance level is .05 when $\rho = 0$, then the significance level becomes .23 for $\rho = .05$. This large change in significance level for a small change in ρ is explained by the factor given for the Student t-distribution in Table I. This shows that test results which appear to be "significant" under the assumption of randomness are not necessarily "significant" when correlation is present, even though the amount of correlation may be small. The effect of correlation on the

TABLE I

Quantity	Distribution For Random Sample	Factor Multiplying Statistic for Correlated "Samples"
$\frac{(\bar{x} - \mu)\sqrt{n(n-1)}}{S} = \frac{(\bar{x} - \mu)\sqrt{n(n-1)}}{\sqrt{\sum_1^n (x_i - \bar{x})^2}}$	Student t-distribution $g_{n-1}(t)\,dt$	$\sqrt{\frac{1-\rho}{1+(n-1)\rho}}$
$\frac{S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_1^n (x_i - \bar{x})^2$	χ^2-distribution $f_{n-1}(\chi^2)\,d\chi^2$	$\frac{1}{1-\rho}$
$\frac{\sigma'^2 S^2}{\sigma^2 S'^2} = \frac{\sigma'^2 \sum_1^n (x_i - \bar{x})^2}{\sigma^2 \sum_1^m (y_\alpha - \bar{y})^2}$	Snedecor F-distribution $h_{n-1,m-1}(F)\,dF$	$\frac{1-\rho'}{1-\rho}$

χ^2 and Snedecor F tests is not as great as for the Student t-test as can be seen from the factors given for the χ^2 and Snedecor F distributions in Table I.

3. Effect of intraclass correlation. The relations stated in Table I will now be used to investigate the effect of intraclass correlation on the confidence coefficients and significance levels of several common types of confidence intervals and significance tests which were derived under the assumption of random samples. The confidence intervals and significance tests considered are listed in Table II, where S^2 and S'^2 are defined in Table I. These particular confidence intervals and significance tests have the property that if α is the confidence coefficient of the confidence interval listed for a given statistic, then $1 - \alpha$ is the significance level of the significance test listed for that statistic, this relation holding whether random samples or correlated "samples" are considered. For this reason the tables given in this section will be limited to confidence coefficients; the corresponding significance levels can be obtained by using the above relation.

a. Student t-distribution. If a random sample of size n is drawn from a normal population with mean μ and variance σ^2 (denoted by $N(\mu, \sigma^2)$), a confidence interval for μ with confidence coefficient ϵ is given in Table II. If the n values form a correlated "sample", however, it follows from Table I that the corresponding confidence interval with coefficient ϵ is

$$\bar{x} - t_\epsilon S\sqrt{\frac{1 + (n-1)\rho}{n(n-1)(1-\rho)}} \leqq \mu \leqq \bar{x} + t_\epsilon S\sqrt{\frac{1 + (n-1)\rho}{n(n-1)(1-\rho)}}.$$

TABLE II

Statistic	Parameter Examined	Confidence Interval (Confidence Coefficient ϵ)	Significance Test (Significance Level = $1 - \epsilon$)	Definitions of Constants
t	μ	$\bar{x} - \frac{t_\epsilon S}{\sqrt{n(n-1)}} \leqq \mu \leqq \bar{x} + \frac{t_\epsilon S}{\sqrt{n(n-1)}}$	$\mid \bar{x} - \mu \mid \leqq t_\epsilon S/\sqrt{n(n-1)}$	$\int_{-t_\epsilon}^{t_\epsilon} g_{n-1}(t)\, dt = \epsilon$
χ^2	σ^2	$O \leqq \sigma^2 \leqq S^2/\chi_\epsilon^2$	$\frac{S^2}{\sigma^2} \leqq 1/\chi_\epsilon^2$	$\int_{\chi_\epsilon^2}^{\infty} f_{n-1}(\chi^2)\, d\chi^2 = \epsilon$
F	$\frac{\sigma^2}{\sigma'^2}$	$O \leqq \sigma^2/\sigma'^2 \leqq S^2/S'^2 F_\epsilon$	$\frac{\sigma^2 S'^2}{\sigma'^2 S^1} \leqq 1/F_\epsilon$	$\int_{F_\epsilon}^{\infty} h_{n-1,m-1}(F)\, dF = \epsilon$

The confidence interval given in Table II can be rewritten as

$$\bar{x} - t_\alpha S\sqrt{\frac{1 + (n-1)\rho}{n(n-1)(1-\rho)}} \leqq \mu \leqq \bar{x} + t_\alpha S\sqrt{\frac{1 + (n-1)\rho}{n(n-1)(1-\rho)}},$$

where

$$t_\alpha = t_\epsilon \sqrt{\frac{1-\rho}{1 + (n-1)\rho}}.$$

Hence if $\rho < 0$, $\alpha > \epsilon$ and the confidence coefficient of the confidence interval in Table II is greater than ϵ. This means that the significance level of the corresponding significance test listed in Table II would be less than $1 - \epsilon$ so that any test result which would be significant for a random sample would also be significant for a correlated "sample" for which $\rho < 0$. If $\rho > 0$, however, $\epsilon > \alpha$ and the significance level of the test would be greater than $1 - \epsilon$. Thus a test result which would be significant for a random sample need no longer be when $\rho > 0$. The effect of positive values of ρ upon the confidence coefficient $\alpha = \alpha_t(\rho, n)$ of the confidence interval of Table II is given in Table III for the cases $\epsilon = .95$ and $.99$. Confidence intervals with unequal tails can be treated in a similar manner. It is thus seen that the effect of correlation on the confidence coefficient increases with the sample size n, and that even a very small amount of correlation can cause a large change in α. For example, for samples of size 16 a correlation of $\rho = .05$ will change the significance level from .05 to .135; for samples of size 32 a correlation of $\rho = .05$ will change the significance level from .01 to .102, and from .05 to .23.

Confidence intervals for $\mu - \mu'$ are given by Theorem 5 of section 4. It is to be observed that if $\rho = \rho' = \rho''$ and $\sigma = \sigma'$ the confidence coefficients are independent of ρ and σ. If $m = n, \rho = \rho', \sigma = \sigma', \rho'' = 0$, however, the confidence coefficients of the confidence intervals for $\mu - \mu'$ have the values $\alpha = \alpha_t(\rho, n)$ given in Table III.

TABLE III

Values of $\alpha_t(\rho, n)$

n \ ρ	0	.05	.1	.2	.3	.4	.5
4	.99 .95		.983 .921	.974 .890	.961 .855	.944 .805	.920 .744
8	.99 .95		.959 .865	.913 .767	.853 .620	.790	
16	.99 .95	.865	.903 .74	.795 .64	.690 .54	.600	.515
32	.99 .95	.898 .77	.79 .68	.63			
64	.99	.79					
128	.99	.68					

b. χ^2-distribution. If a random sample of size n is drawn from $N(\mu, \sigma^2)$, a confidence interval for σ^2 with coefficient ϵ is given in Table II. If the n values form a correlated "sample", it follows from Table I that the corresponding confidence interval with coefficient ϵ is

$$0 \leqq \sigma^2 \leqq S^2/\chi^2_\epsilon(1 - \rho).$$

The confidence interval in Table II can be rewritten as

$$0 \leqq \sigma^2 \leqq S^2/\chi^2_\alpha(1 - \rho),$$

where

$$\chi^2_\alpha = \chi^2_\epsilon/(1 - \rho).$$

Hence if $\rho < 0$, $\alpha > \epsilon$ and the significance level of the significance test given in Table II is less than $1 - \epsilon$. If $\rho > 0$, the significance level of the test is greater than $1 - \epsilon$. The effect of positive values of ρ upon the confidence coefficient $\alpha = \alpha_{\chi^2}(\rho, n)$ of the confidence interval listed in Table II is given in Table IV for $\epsilon = .95$ and $.99$. Cases in which the lower limit of the confidence interval is not zero can be treated in a similar manner. Table IV shows that the confidence coefficient $\alpha = \alpha_{\chi^2}(\rho, n)$ decreases with the sample size n for a fixed value of ρ. Although the effect of correlation for the χ^2-distribution is not as great as for the Student t-distribution, it does cause a noticeable change in α. For example, for samples of size 16 the significance level of the test in Table II is changed from .05 to .081 if $\rho = .1$ and from .05 to .13 if $\rho = .2$. For samples of size 32 the significance level is changed from .05 to .10 for $\rho = .1$ and from .05 to .19 for $\rho = .2$.

c. Snedecor f-distribution. If two random samples, one of size n (denoted by x's) and the other of size m (denoted by y's), are drawn from $N(\mu, \sigma^2)$ and $N(\mu', \sigma'^2)$ respectively, a confidence interval for σ^2/σ'^2 with coefficient ϵ

TABLE IV

Values of $\alpha_{\chi^2}(\rho, n)$

n \ ρ	0	.1	.2	.3	.4	5.
4	.99	.988	.986	.983	.979	.971
	.95	.941	.930	.918	.900	.872
16	.99	.982	.966	.941	.890	.790
	.95	.919	.87	.79	.67	.49
32	.99	.975	.946	.867	.715	.44
	.95	.90	.81	.64	.38	.17

is given in Table II. If the values form two correlated "samples", however, it follows from Table I that the corresponding confidence interval with coefficient ϵ is

$$0 \leqq \sigma^2/\sigma'^2 \leqq \frac{S^2(1 - \rho')}{S'^2(1 - \rho)} \Big/ F_\epsilon .$$

The confidence interval in Table II can be restated as

$$0 \leqq \sigma^2/\sigma'^2 \leqq \frac{S^2(1 - \rho')}{S'^2(1 - \rho)} \Big/ F_\alpha ,$$

where

$$F_\alpha = F_\epsilon(1 - \rho')/(1 - \rho).$$

Thus if $\rho = \rho'$, $\alpha = \epsilon$ and the significance level of the significance test given in Table II remains equal to $1 - \epsilon$. If $(1 - \rho')/(1 - \rho) < 1$, $\alpha > \epsilon$ and the significance level is less than $1 - \epsilon$. If $(1 - \rho')/(1 - \rho) > 1$, however, $\alpha < \epsilon$

and the significance level is greater than $1 - \epsilon$. Values of the confidence coefficient $\alpha = \alpha_F\left(\frac{1-\rho'}{1-\rho}, n, m\right)$ of the confidence interval listed in Table II are given in Table V for $\epsilon = .95$ and $.99$. Cases in which the lower limit of the confidence interval is not zero can be treated in a manner similar to that given above. Table V indicates that the effect of correlation on the confidence coefficient is not as great for $n < m$ as for $n > m$. For example, if $n = 4, m = 32$,

TABLE V

Values of $\alpha_F\left(\frac{1-\rho'}{1-\rho}, n, m\right)$

n	m \ $\frac{1-\rho'}{1-\rho}$	1	1.25	1.5	2.0
4	4	.99 .95	.987 .933	.983 .916	.975 .880
16	4	.99 .95	.978 .912	.962 .869	.917 .778
32	4	.99 .95	.975 .906	.952 .858	.896 .753
4	16	.99 .95	.987 .933	.985 .914	.977 .875
16	16	.99 .95	.973 .892	.945 .817	.858 .637
32	16	.99 .95	.919 .869	.837 .763	.628 .518
4	32	.99 .95	.987 .931	.985 .913	.977 .874
32	32	.99 .95	.960 .850	.893 .707	.675 .400

$\frac{1-\rho'}{1-\rho} = 1.25$, the significance level of the significance test given in Table II is only changed from .05 to .069, if $\frac{1-\rho'}{1-\rho} = 1.5$ from .05 to .087. If $n = 32, m = 4$, $\frac{1-\rho'}{1-\rho} = 1.25$, however, the significance level is changed from .05 to .094, if $\frac{1-\rho'}{1-\rho} = 1.5$ from .05 to .142. Also it is seen that for fixed $\frac{1-\rho'}{1-\rho}$, the effect of intraclass correlation increases with both n and m.

4. Analysis. This section contains derivations of the relations stated in the first three sections. The method used in these derivations is similar to that used in one approach to the analysis of variance and consists essentially in expressing each variable as the sum of two quantities, one of which is the same for each variable and the other of which is different for each variable.

Let $x_1, \cdots, x_n$ represent a correlated "sample", that is, have a normal multivariate distribution for which

$$
\begin{aligned}
&E(x_i) = \mu, && (i = 1, \cdots, n)\\
(1)\qquad &E[(x_i - \mu)^2] = \sigma^2 &&\\
&E[(x_i - \mu)(x_j - \mu)] = \rho\sigma^2, && (i \neq j = 1, \cdots, n).
\end{aligned}
$$

Write the x_i, $(i = 1, \cdots, n)$, in the form

$$x_i = \eta + \lambda\bar{\xi} + \xi_i,$$

where $\bar{\xi} = \sum_1^n \xi_i/n$ and $\eta, \xi_1, \cdots, \xi_n$ are independently distributed, η according to $N(\mu, \sigma_\eta^2)$ and the ξ_i according to $N(0, \sigma_\xi^2)$. The values of λ, σ_η^2 and σ_ξ^2 are chosen so that the $x_i = \eta + \lambda\bar{\xi} + \xi_i$ satisfy (1). It is easily proved that it is always possible to choose λ, σ_η^2 and σ_ξ^2 so that (1) are satisfied. It is to be remembered that $\rho \geqq -1/(n - 1)$ for intraclass correlation. From relations (1) and $x_i = \eta + \lambda\bar{\xi} + \xi_i$ it follows that

$$(2)\qquad E(\xi_i^2) = \sigma^2(1 - \rho), \qquad (i = 1, \cdots, n).$$

THEOREM 1. *The quantity* $\frac{1}{\sigma^2(1 - \rho)} \sum_1^n (x_i - \bar{x})^2$ *has a* χ^2*-distribution with* $n - 1$ *degrees of freedom and is distributed independently of* $\bar{x}$.

PROOF. Since the ξ_i are independently distributed according to the same normal distribution with zero mean, it follows from (2) that

$$\frac{1}{E(\xi_i^2)} \sum_1^n (\xi_i - \bar{\xi})^2 = \frac{1}{\sigma^2(1 - \rho)} \sum_1^n (x_i - \bar{x})^2$$

has a χ^2-distribution with $n - 1$ degrees of freedom and is distributed independently of $\bar{x} = \eta + (1 + \lambda)\bar{\xi}$.

THEOREM 2. $\frac{(\bar{x} - \mu)\sqrt{n(n - 1)}}{\sqrt{1 + (n - 1)\rho}} \bigg/ \sqrt{\sum_1^n \frac{(x_i - \bar{x})^2}{1 - \rho}}$ *has a Student t-distribution with* $n - 1$ *degrees of freedom.*

PROOF. It is easily seen from elementary considerations that $\frac{(\bar{x} - \mu)\sqrt{n}}{\sigma\sqrt{1 + (n - 1)\rho}}$ has the distribution $N(0, 1)$. Theorem 2 is then an immediate consequence of Theorem 1.

Up to this point a single correlated "sample" of size n has been considered. The next part of the analysis, however, will be concerned with properties which arise from the consideration of two correlated "samples."

Let $x_1, \cdots, x_n, y_1, \cdots, y_m$ have a joint normal multivariate distribution such that

$$
\begin{aligned}
&E(x_i) = \mu, && (i = 1, \cdots, n)\\
&E(y_\alpha) = \mu', && (\alpha = 1, \cdots, m)\\
&E[(x_i - \mu)^2] = \sigma^2 &&\\
&E[(y_\alpha - \mu')^2] = \sigma'^2 && \\
&E[(x_i - \mu)(x_j - \mu)] = \rho\sigma^2, && (i \neq j = 1, \cdots, n)\\
&E[(y_\alpha - \mu')(y_\beta - \mu')] = \rho'\sigma'^2, && (\alpha \neq \beta = 1, \cdots, m)\\
&E[(x_i - \mu)(y_\alpha - \mu')] = \rho''\sigma\sigma'. &&
\end{aligned}
\tag{3}
$$

Write the x_i and y_α in the form

$$
\begin{aligned}
x_i &= \eta + \lambda_1\bar{\xi} + \lambda_2\bar{\xi}' + \xi_i\\
y_\alpha &= \eta' + \lambda_1'\bar{\xi} + \lambda_2'\bar{\xi}' + \xi_\alpha',
\end{aligned}
\tag{4}
$$

where $\bar{\xi}' = \sum_1^m \xi_\alpha'/m$ and $\eta, \eta', \xi_1, \cdots, \xi_n, \xi_1', \cdots, \xi_m'$ are independently distributed, η according to $N(\mu, \sigma_\eta^2)$, η' according to $N(\mu', \sigma_\eta'^2)$, the ξ_i according to $N(0, \sigma_\xi^2)$, and the ξ_α' according to $N(0, \sigma_\xi'^2)$. The quantities $\lambda_1, \lambda_2, \lambda_1', \lambda_2'$, $\sigma_\eta^2, \sigma_\eta'^2, \sigma_\xi', \sigma_\xi'^2$ are chosen so that the x_i and y_α satisfy (3). It is easily verified that it is always possible to choose these quantities so that the x_i and y_α constructed in this fashion satisfy (3). In addition it follows from (3) and (4) that

$$
\begin{aligned}
E(\xi_i^2) &= \sigma^2(1 - \rho)\\
E(\xi_\alpha'^2) &= \sigma'^2(1 - \rho').
\end{aligned}
\tag{5}
$$

THEOREM 3. $\frac{1}{\sigma^2(1 - \rho)} \sum_1^n (x_i - \bar{x})^2$ *and* $\frac{1}{\sigma'^2(1 - \rho')} \sum_1^m (y_\alpha - \bar{y})^2$ *have* χ^2-*distributions with* $n - 1$ *and* $m - 1$ *degrees of freedom respectively, and are distributed independently of each other and of* $\bar{x}$ *and* $\bar{y}$.

PROOF. From Theorem 1 and (5) it follows that $\frac{1}{\sigma^2(1 - \rho)} \sum_1^n (x_i - \bar{x})^2$ and $\frac{1}{\sigma'^2(1 - \rho')} \sum_1^m (y_\alpha - \bar{y})^2$ have χ^2-distributions with $n - 1$ and $m - 1$ degrees of freedom respectively. That they are distributed independently of each other and of both $\bar{x}$ and $\bar{y}$ follows from (4).

THEOREM 4. $\dfrac{\sigma'^2(1 - \rho') \sum_1^n (x_i - \bar{x})^2}{\sigma^2(1 - \rho) \sum_1^m (y_\alpha - \bar{y})^2}$ *is distributed according to the Snedecor* F-*distribution* $h_{n-1,\, m-1}(F)dF$.

PROOF. This follows from Theorem 3.

THEOREM 5.

$$\frac{[(\bar{x} - \bar{y}) - (\mu - \mu')]\sqrt{n + m - 2}}{\sigma_1} \Bigg/ \sqrt{\frac{\sum_1^n (x_i - \bar{x})^2}{\sigma^2(1 - \rho)} + \frac{\sum_1^m (y_\alpha - \bar{y})^2}{\sigma'^2(1 - \rho')}}$$

where

$$\sigma_1^2 = \frac{\sigma^2}{n}[1 + (n - 1)\rho] + \frac{\sigma'^2}{m}[1 + (m - 1)\rho'] - 2\rho''\sigma\sigma',$$

has a Student t-distribution with $n + m - 2$ degrees of freedom.

PROOF. It is easily seen from elementary considerations that $\frac{1}{\sigma_1}[(\bar{x} - \bar{y}) - (\mu - \mu')]$ has the distribution $N(0, 1)$. Theorem 5 then follows from Theorem 3.

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