ON THE CONVERGENCE OF SEQUENCES OF MOMENT GENERATING FUNCTIONS

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1. Summary. The purpose of this paper is to give a few theorems concerning the reciprocal relation between the convergence of a sequence of distribution functions and the convergence of the corresponding sequence of their moment generating functions.

The paper consists of two parts. In the first part the univariate case is discussed. The content of this part is closely related to that of a recent paper by J. H. Curtiss [1, p. 430–433], but the results are of a somewhat more general nature, and the methods of proofs are different and do not make use of the theory of a complex variable. The second part deals with the multivariate case which, as far as the author knows, has not been treated before with proofs in as complete and rigorous a way.

In both the univariate and multivariate cases the proofs are based on the well known Helly selection principle [2, p. 26] for bounded sequences of monotonic functions.

2. The univariate case. Let X be a random variable and $F(x)$ its distribution function. That is, for any real x, $F(x) = P\{X \leq x\}$, where $P\{X \leq x\}$ denotes the probability of the event $X \leq x$. The function

$$\varphi(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tX}\, dF(x),$$

in which the integral is taken in the Stieltjes-Riemann sense and is assumed to converge in some neighborhood of the origin, is called the moment generating function of X (or of $F(x)$).

Henceforth we use the abbreviations d.f. and m.g.f. for the terms distribution function and moment generating function respectively. The variable t will be always real.

Theorem 1. *Let $\{F_n(x)\}$ be a sequence of d.f.'s. Let $M(x)$ for any fixed non-negative x be the least upper bound of the sequence $\{F_n(-x) + 1 - F_n(x)\}$. If the sequence $\{F_n(x)\}$ converges on an everywhere dense set of points on the x-axis, and if there exists a positive number α such that for any fixed t in the interval $|t| < \alpha$*

$$\lim_{x \to +\infty} e^{|t|x} M(x) = 0, \tag{1}$$

then:

(a) *there exists a d.f. $F(x)$ such that $\lim_{n\to\infty} F_n(x) = F(x)$ at each point of continuity of of $F(x)$;*

(b) *the m.g.f.'s of $F(x)$ and $F_n(x)$, say $\varphi(t)$ and $\varphi_n(t)$ exist for $|t| < \alpha$;*

(c) $\lim_{n\to\infty} \varphi_n(t) = \varphi(t)$ *for $|t| < \alpha$ and uniformly in each interval $|t| \leq \beta < \alpha$.*

To prove (a), it may be noticed that there exists a function $F(x)$, non-decreasing and continuous on the right, such that $\lim_{n\to\infty} F_n(x) = F(x)$ at each point of concontinuity of $F(x)$. But $F(x)$ must be a distribution function. Indeed, we have for $x > 0$

$$F(-x) + 1 - F(x) \leq M(x-). \tag{2}$$

Now from (1), putting $t = 0$, we find that $M(x)$ and consequently $M(x-)$ approach zero as $x \to +\infty$. This proves that $F(-\infty) = 0$ and $F(+\infty) = 1$.

To prove (b), we notice first that the integral

$$\varphi_n(t) = \int_{-\infty}^{+\infty} e^{xt}\, dF_n(x) \qquad (n = 1, 2, \cdots),$$

is convergent for $|t| < \alpha$. This follows immediately from (1) by applying the method of integration by parts to the integrals

$$\int_0^N e^{xt}\, dF_n(x) \quad \text{and} \quad \int_{-N}^0 e^{xt}\, dF_n(x),$$

which for any t in the interval $|t| < \alpha$ will be seen to be bounded for all values of N. By the same argument, the relation $\lim_{x\to+\infty} M(x-)e^{|t|x} = 0$, $|t| < \alpha$, which can be easily deduced from (1), together with (2) imply that the integral representing $\varphi(t)$ is convergent for $|t| < \alpha$.

Let now β be a positive number less than α and let γ be such that $\beta < \gamma < \alpha$. Let M_γ be the least upper bound of $M(x)e^{\gamma x}$ for $x > 0$. Using the method of integration by parts and applying (1) we have for $|t| \leq \beta$

$$\begin{aligned} \int_N^{+\infty} e^{xt}\, dF_n(x) &= [1 - F_n(N)]\, e^{Nt} + t \int_N^{+\infty} e^{xt}[1 - F_n(x)]\, dx \\ &\leq M(N)e^{N\beta} + M_\gamma \beta \frac{e^{N(\beta-\gamma)}}{\gamma - \beta}. \end{aligned} \tag{3}$$

We could prove easily that the same inequality is true for the integrals

$$\int_{-\infty}^{-N} e^{xt}\, dF_n(x), \qquad \int_N^{+\infty} e^{xt}\, dF(x), \qquad \int_{-\infty}^{-N} e^{xt}\, dF(x).$$

Now let ϵ be any positive number. Because of (3), we have

$$\int_{|x|>N_0} e^{xt}\, dF_n(x) < \epsilon, \qquad \int_{|x|>N_0} e^{xt}\, dF(x) < \epsilon, \tag{4}$$

for a sufficiently great number N_0, and uniformly with respect to n and t, when $|t| \leq \beta$. Clearly, N_0 can be so chosen that $F(x)$ is continuous for $x = \pm N_0$. Then

$$\lim_{n\to\infty} \int_{-N_0}^{N_0} e^{xt}\, dF_n(x) = \int_{-N_0}^{N_0} e^{xt}\, dF(x), \tag{5}$$

uniformly for $|t| \leq \beta$.

The relations (4) and (5) prove that $\varphi_n(t) \to \varphi(t)$ as $n \to \infty$, uniformly for $|t| \leq \beta$. But β can be chosen as near to α as we please; thus (c) is proved.

THEOREM 2. *Let* $\{F_n(x)\}$ *be a sequence of d.f.'s and* $\{\varphi_n(t)\}$ *the corresponding sequence of m.g.f.'s. If* $\varphi_n(t)$ *exists for* $|t| < \alpha$, *and if there exists a finite valued function* $\varphi(t)$ *defined for* $|t| < \alpha$, *such that* $\lim_{n\to\infty} \varphi_n(t) = \varphi(t)$ *for* $|t| < \alpha$, *then*

(a) $$\lim_{x\to+\infty} M(x)e^{|t|x} = 0 \text{ for } |t| < \alpha;$$

(b) *there exists a d.f.* $F(x)$ *such that* $\lim_{n\to\infty} F_n(x) = F(x)$ *at each point of continuity of* $F(x)$

(c) *the m.g.f. of* $F(x)$ *exists for* $|t| < \alpha$ *and is identically equal to* $\varphi(t)$ *in this interval.*

(d) $\lim_{n\to\infty} \varphi_n(t) = \varphi(t)$ *uniformly in each interval* $|t| \leq \beta < \alpha$.

To prove (a), let t be a number in the interval $|t| < \alpha$, and let β be chosen so that $|t| < \beta < \alpha$. Then, for $x \geq 0$, we have

$$\begin{aligned} F_n(-x) + 1 - F_n(x) &= \int_{-\infty}^{-x} dF_n(u) + \int_x^{+\infty} dF_n(u) \\ &\leq e^{-\beta x} \int_{-\infty}^{-x} e^{-\beta u}\, dF_n(u) + e^{-\beta x} \int_x^{+\infty} e^{\beta u}\, dF_n(u) \\ &\leq e^{-\beta x}[\varphi_n(-\beta) + \varphi_n(\beta)]. \end{aligned}$$

Consequently

$$M(x)e^{|t|x} \leq e^{(|t|-\beta)x} \operatorname*{l.u.b.}_n \{\varphi_n(-\beta) + \varphi_n(\beta)\},$$

and since the sequences $\{\varphi_n(-\beta)\}$ and $\{\varphi_n(\beta)\}$ are convergent, and therefore bounded, it follows that $M(x)e^{|t|x}$ approaches zero as $x \to +\infty$.

To prove (b) we may notice that by the Helly selection principle we can choose a subsequence $\{F_{n_k}(x)\}$ which is convergent to some non-decreasing function $F(x)$, at each point of continuity of $F(x)$. Now the Theorem 1 together with (a) imply that $F(x)$ is a d.f. and that the limit of the subsequence $\{\varphi_{n_k}(t)\}$, namely $\varphi(t)$, must be identical, for $|t| < \alpha$, with the m.g.f. of $F(x)$. By the uniqueness property of a m.g.f. we know that $F(x)$ is uniquely determined by $\varphi(t)$, and therefore it follows that every convergent subsequence of $\{F_n(x)\}$ approaches the same limit $F(x)$ at each point of continuity of $F(x)$. This is, however, equivalent to the statement that the sequence $\{F_n(x)\}$ itself converges to $F(x)$ at each point of continuity of $F(x)$. Thus (b) is proved. We see at once that (c) and (d) follow immediately from the Theorem 1.

Theorem 2 is of course similar to the Theorem 3 in the paper of Curtiss [1, p. 432]. The proof of (a), however, is not contained in his paper. From the Theorems 1 and 2 there follows immediately

THEOREM 3. *Let* $\{F_n(x)\}$ *be a sequence of d.f.'s, and let* $\{\varphi_n(t)\}$ *be the correspond-*

ing sequence of m.g.f.'s, which are all assumed to exist for $|t| < \alpha$. *The necessary and sufficient conditions for the convergence of* $\{\varphi_n(t)\}$ *in the interval* $|t| < \alpha$, *are:*

(a) $\lim_{n \to +\infty} M(x)e^{|t|x} = 0, \qquad |t| < \alpha$

(b) *the sequence* $\{F_n(x)\}$ *converges to a d.f.* $F(x)$ *at each point of continuity of* $F(x)$. *Further, the m.g.f. of* $F(x)$ *exists for* $|t| < \alpha$ *and is equal in this interval to the limit of the sequence* $\{\varphi_n(t)\}$.

In his paper Curtiss gives an example of a sequence $\{F_n(x)\}$ of d.f.'s which converges to a d.f. $F(x)$, while the corresponding sequence $\{\varphi_n(t)\}$ of m.g.f.'s does not converge to the m.g.f. $\varphi(t)$ of the d.f. $F(x)$, though both $\varphi_n(t)$, $(n = 1, 2, \cdots)$, and $\varphi(t)$ exist for all t. It may be easily proved by the direct method that in the case considered the condition (a) of the Theorem 3 is not satisfied.

It is perhaps worth while to notice that the condition (a) of the Theorem 3 may be expressed also as follows:

$$\overline{\lim_{x \to +\infty}}\, x^{-1} \log M(x) \leq -\alpha.$$

3. The multivariate case. For the sake of simplicity we shall consider here the bivariate case only. The results obtained in this chapter, can be, however, easily extended to the case when d.f.'s and m.g.f.'s are defined in the Euclidean space of any finite number of dimensions.

Let (X_1, X_2) be a random vector variable in the two-dimensional Euclidean space, and let $F(x_1, x_2)$ be its d.f. That is, for any real numbers x_1 and x_2,

$$F(x_1, x_2) = P\{X_1 \leq x_1, X_2 \leq x_2\}.$$

Let

$$F_1(x_1) = P\{X_1 \leq x_1\} = F(x_1, +\infty),$$

$$F_2(x_2) = P\{X_2 \leq x_2\} = F(+\infty, x_2);$$

then $F_1(x_1)$ and $F_2(x_2)$ are called the marginal d.f.'s of X_1 and X_2 respectively. The m.g.f.'s of the d.f.'s $F(x_1, x_2)$, $F_1(x_1)$ and $F_2(x_2)$ are defined by the equations:

$$\varphi(t_1, t_2) = E(e^{X_1 t_1 + X_2 t_2}) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{x_1 t_1 + x_2 t_2}\, dF(x_1, x_2)$$

$$\varphi_i(t_i)) = E(e^{X_i t_i}) = \int_{-\infty}^{+\infty} e^{x_i t_i}\, dF_i(x_i), \qquad (i = 1, 2),$$

in which the integrals are assumed to converge in some neighborhood of the origin. It is easy to see that $\varphi_1(t_1) = \varphi(t_1, 0)$ and $\varphi_2(t_2) = \varphi(0, t_2)$.

THEOREM 4. *Let* $\varphi(t_1, t_2)$ *and* $\varphi^*(t_1, t_2)$ *be the m.g.f.'s of d.f.'s* $F(x_1, x_2)$ *and* $F^*(x_1, x_2)$ *respectively. If* $\varphi(t_1, t_2)$ *and* $\varphi^*(t_1, t_2)$ *exist and are equal in some neighborhood of the origin* $|t_i| < \alpha_i$, $(i = 1, 2)$, *then* $F(x_1, x_2) = F^*(x_1, x_2)$ *identically.*

To prove this theorem, let us introduce two random vector variables (X_1, X_2)

and (X_1^*, X_2^*) of which the d.f.'s are respectively F and F^*. Consider now two random variables

$$Z = X_1t_1 + X_2t_2, \qquad Z^* = X_1^*t_1 + X_2^*t_2,$$

where t_1 and t_2 denote two real numbers not both zero. If $\varphi(t)$ and $\varphi^*(t)$ are respectively the m.g.f.'s of Z and Z^*, we have

$$\varphi(t) = \varphi(tt_1, tt_2), \qquad \varphi^*(t) = \varphi^*(tt_1, tt_2).$$

Consequently $\varphi(t) = \varphi^*(t)$ provided that $|tt_i| < \alpha_i$, $(i = 1, 2)$. It follows from the uniqueness property of the m.g.f. in the univariate case that the d.f.'s of Z and Z^* must be identical. Now, according to a theorem due to Cramér [3, p. 105], if the d.f.'s of Z and Z^* coincide for all pairs of values (t_1, t_2) such that $|t_1| + |t_2| \neq 0$, the d.f.'s F and F^* must be identical. It may be worth while to reproduce here Cramér's proof. Let $\psi(t_1, t_2) = E(e^{i(X_1t_1+X_2t_2)})$ and $\psi^*(t_1, t_2) = E(e^{i(X_1^*t_1+X_2^*t_2)})$ be the characteristic functions of F and F^* respectively. Then $\psi(tt_1, tt_2)$ and $\psi^*(tt_1, tt_2)$ are the characteristic functions of Z and Z^* respectively. Since Z and Z^* have the same d.f.'s, it follows that $\psi(tt_1, tt_2) = \psi^*(tt_1, tt_2)$ for all values of t. Putting $t = 1$, we find that $\psi(t_1, t_2) = \psi^*(t_1, t_2)$ if $|t_1| + |t_2| \neq 0$. For $t_1 = t_2 = 0$, $\psi(0, 0) = \psi^*(0, 0) = 1$. Therefore $\psi(t_1, t_2) = \psi^*(t_1, t_2)$ identically, and since the characteristic function uniquely determines the d.f., it follows that the d.f. F and F^* are identical.

THEOREM 5. *Let $\{F_n(x_1, x_2)\}$ be a sequence of d.f.'s. Let $F_{1n}(x_1)$ and $F_{2n}(x_2)$ be respectively the marginal d.f.'s determined by $F_n(x_1, x_2)$. Let*

$$M_i(x_i) = \underset{n}{\text{l.u.b.}} \{F_{in}(-x_i) + 1 - F_{in}(x_i)\}$$

where $x_i \geq 0$, $(i = 1, 2)$. If there exist positive numbers α_1 and α_2 such that for $|t_i| < \alpha_i$

$$\lim_{x_i \to +\infty} M_i(x_i)e^{|t_i|x_i} = 0, \qquad (i = 1, 2), \tag{6}$$

and if $\{F_n(x_1, x_2)\}$ converges on an everywhere dense set on the (x_1, x_2) plane, then:

(a) *there exists a d.f. $F(x_1, x_2)$ such that $\lim_{n\to\infty} F_n(x_1, x_2) = F(x_1, x_2)$ at each point of continuity of $F(x_1, x_2)$,*

(b) *there exist two positive numbers δ_1 and δ_2, $\delta_i < \alpha_i$, such that the m.g.f.'s of $F(x_1, x_2)$ and $F_n(x_1, x_2)$, say $\varphi(t_1, t_2)$ and $\varphi_n(t_1, t_2)$, exist for $|t_i| < \delta_i$, $(i = 1, 2)$,*

(c) *$\lim_{n\to\infty} \varphi_n(t_1, t_2) = \varphi(t_1, t_2)$ for $|t_i| < \delta_i$, and uniformly in each two-dimensional interval $|t_i| \leq \beta_i < \delta_i$, $(i = 1, 2)$.*

To prove (a), we notice that there obviously exists a function $F(x_1, x_2)$, continous on the right with respect to each variable, satisfying the relation

$$\Delta^2F(x_1, x_2) = F(x_1'', x_2'') + F(x_1', x_2') - F(x_1', x_2'') - F(x_1'', x_2') \geq 0$$

for $x_1' \leq x_1''$, $x_2' \leq x_2''$, and such that

$$\lim_{n \to \infty} F_n(x_1, x_2) = F(x_1, x_2) \tag{7}$$

at each point of continuity of $F(x_1, x_2)$. We shall prove that $F(x_1, x_2)$ is a d.f. In fact, it is easy to see that we have for $x_i > 0$, $(i = 1, 2)$,

$$F(-x_1, -x_2) \leq F(-x_1, x_2) \leq M_1(x_1-), \quad F(x_1, -x_2) \leq M_2(x_2-), \tag{8}$$
$$1 - F(x_1, x_2) \leq M_1(x_1) + M_2(x_2).$$

Now, according to (6), $\lim_{x_i \to +\infty} M_i(x_i-) = \lim_{x_i \to +\infty} M_i(x_i) = 0$, $(i = 1, 2)$, therefore it follows from (8) that $F(-\infty, -\infty) = F(-\infty, x_2) = F(x_1, -\infty) = 0$ and $F(+\infty, +\infty) = 1$, which proves that $F(x_1, x_2)$ is a d.f.

To prove (b), let $\varphi_{in}(t_i)$ be the m.g.f. of the d.f. $F_{in}(x_i)$, $(i = 1, 2)$. Let $F_1(x_1)$ and $F_2(x_2)$ be the marginal d.f.'s determined by $F(x_1, x_2)$ and let $\varphi_i(t_i)$ be the m.g.f. of $F_i(x_i)$, $(i = 1, 2)$.

Now let $N' > N > 0$ and

$$R_n(N, N', t_1, t_2) = \int_{-N'}^{N'} \int_{-N'}^{N'} e^{x_1 t_1 + x_2 t_2} \, dF_n(x_1, x_2) - \int_{-N}^{N} \int_{-N}^{N} e^{x_1 t_1 + x_2 t_2} \, dF_n(x_1, x_2)$$
$$= \int_{N}^{N'} \int_{-N'}^{N} + \int_{-N}^{N'} \int_{N}^{N'} + \int_{-N'}^{-N} \int_{-N}^{N'}$$
$$+ \int_{-N'}^{N} \int_{-N'}^{-N} e^{x_1 t_1 + x_2 t_2} \, dF_n(x_1, x_2) = I_1 + I_2 + I_3 + I_4 .$$

Applying the Schwartz inequality to I_1, we find

$$I_1 \leq \left(\int_{N}^{N'} \int_{-N'}^{N} e^{2x_1 t_1} \, dF_n \right)^{\frac{1}{2}} \left(\int_{N}^{N'} \int_{-N'}^{N} e^{2x_2 t_2} \, dF_n \right)^{\frac{1}{2}}. \tag{9}$$

But

$$\int_{N}^{N'} \int_{-N'}^{N} e^{2x_1 t_1} \, dF_n(x_1, x_2) \leq \int_{N}^{N'} e^{2x_1 t_1} \, dF_{1n}(x_1), \tag{10}$$

and similarly

$$\int_{N}^{N'} \int_{-N'}^{N} e^{2x_2 t_2} \, dF_n(x_1, x_2) \leq \int_{-N'}^{N} e^{2x_2 t_2} \, dF_{2n}(x_2). \tag{11}$$

Let ϵ be any positive number and γ_i a positive number less than α_i, $(i = 1, 2)$. It follows from the proof of the Theorem 1, taking into account (6), that the integrals representing $\varphi_{in}(t_i)$ and $\varphi_i(t_i)$, $(i = 1, 2)$, exist and are uniformly convergent with respect to n and t_i, when $|t_i| \leq \gamma_i$, $(i = 1, 2)$. Consequently we have

$$\int_{|x_i| > N} e^{x_i t} \, dF_{in}(x_i) < \epsilon, \qquad \int_{|x_i| > N} e^{x_i t_i} \, dF_i(x_i) < \epsilon, \qquad (i = 1, 2), \tag{12}$$

uniformly with respect to n and t_i when $|t| \leq \gamma_i$, $(i = 1, 2)$, provided that N is sufficiently large, say $N \geq N_0$. Let us take $\beta_i = \gamma_i/2$, $(i = 1, 2)$. The integrals representing $\varphi_{in}(t_i)$ and $\varphi_i(t_i)$, $(i = 1, 2)$, are obviously uniformly bounded for all n and when $|t_i| \leq \gamma_i$, $(i = 1, 2)$, they are all less than some constant C. Consequently taking into account (9), (10), (11), and (12), we find

$$I_1 < \sqrt{C\epsilon},$$

uniformly with respect to n and t_i when $|t_i| \leq \beta_i$, $(i = 1, 2)$, provided that $N' > N \geq N_0$. Since the same inequality is true for I_2, I_3 and I_4, we have

$$R_n(N, N', t_1, t_2) < 4\sqrt{C\epsilon}, \tag{13}$$

uniformly with respect to n and t_i, when $|t_i| \leq \beta_i$, $(i = 1, 2)$, provided $N' > N \geq N_0$. Hence the integral representing $\varphi_n(t_1, t_2)$ is uniformly convergent for $|t_i| \leq \beta_i$, and consequently convergent for $|t_i| < \alpha_i/2$, $(i = 1, 2)$, since β_i can be chosen as near to $\alpha_i/2$ as we please.

Similarly, using (12), we could find

$$R(N, N', t_1, t_2) < 4\sqrt{C\epsilon}, \qquad |t_i| \leq \beta_i, N' > N \geq N_0 \tag{14}$$

where

$$R(N, N', t_1, t_2) = \int_{-N'}^{N'}\int_{-N'}^{N'} e^{x_1t_1+x_2t_2}\,dF(x_1, x_2) - \int_{-N}^{N}\int_{-N}^{N} e^{x_1t_1+x_2t_2}\,dF(x_1, x_2).$$

This proves, in turn, that the integral representing $\varphi(t_1, t_2)$ is uniformly convergent for $|t_i| \leq \beta_i$ and convergent for $|t_i| < \alpha_i/2$, $(i = 1, 2)$. Thus (b) is proved with $\delta_i = \alpha_i/2$, $(i = 1, 2)$.

To prove (c), let $N' \to +\infty$ and $N = N_0$ in (13) and (14). We obtain

$$R_n(N_0, +\infty, t_1, t_2) \leq 4\sqrt{C\epsilon}, \qquad R(N_0, +\infty, t_1, t_2) \leq 4\sqrt{C\epsilon} \tag{15}$$

uniformly with respect to n and t_i when $|t_i| \leq \beta_i$.

Clearly, N_0 can be chosen so that $F_1(x_1)$ and $F_2(x_2)$ are continuous for $x_1 = x_2 = \pm N_0$. Then

$$\lim_{n\to\infty}\int_{-N_0}^{N_0}\int_{-N_0}^{N_0} e^{x_1t_1+x_2t_2}\,dF_n(x_1, x_2) = \int_{-N_0}^{N_0}\int_{-N_0}^{N_0} e^{x_1t_1+x_2t_2}\,dF(x_1, x_2), \tag{16}$$

uniformly for $|t_i| \leq \beta_i$, $(i = 1, 2)$.

The relations (15) and (16) prove that

$$\lim_{n\to\infty} \varphi_n(t_1, t_2) = \varphi(t_1, t_2),$$

uniformly for $|t_i| \leq \beta_i$, $(i = 1, 2)$. The ordinary convergence obviously holds for $|t_i| < \alpha_i/2$, $(i = 1, 2)$.

It follows from the above proof, which refers to the bivariate case, that we may take $\delta_i = \alpha_i/2$, $(i = 1, 2)$, in (b) and (c).

The existence of the corresponding numbers δ_i, $\delta_i < \alpha_i$, $(i = 1, 2, \cdots, k)$, in the k-variate case can be easily established by the repeated application of the Schwartz inequality.

THEOREM 6. *Let $\varphi_n(t_1, t_2)$, $\varphi_{in}(t_i)$, $F_n(x_1, x_2)$, $F_{in}(x_i)$ and $M_i(x_i)$, $(i = 1, 2)$, have the same meaning as in the Theorem 5. If $\varphi_n(t_1, t_2)$ exist for $|t_i| < \alpha_i$, $(i = 1, 2)$, and if there exists a finite valued function $\varphi(t_1, t_2)$ defined for $|t_i| < \alpha_i$, such that $\lim_{n\to\infty} \varphi_n(t_1, t_2) = \varphi(t_1, t_2)$, $|t_i| < \alpha_i$,*

then

(a) $$\lim_{x_i \to +\infty} M_i(x_i)\, e^{|t_i| x_i} = 0 \qquad \text{for } |t_i| < \alpha_i, \qquad (i = 1, 2),$$

(b) *there exists a d.f. $F(x_1, x_2)$, such that $\lim_{n\to\infty} F_n(x_1, x_2) = F(x_1, x_2)$ at each point of continuity of $F(x_1, x_2)$,*

(c) *the m.g.f. of $F(x_1, x_2)$ exists for $|t_i| < \alpha_i$ and is identically equal to $\varphi(t_1, t_2)$ for $|t_i| < \alpha_i$, $(i = 1, 2)$,*

(d) *$\lim_{n\to\infty} \varphi_n(t_1, t_2) = \varphi(t_1, t_2)$ uniformly for $|t_i| \leq \beta_i < \alpha_i$, $(i = 1, 2)$.*

To prove (a), it is sufficient to notice that $\varphi_{1n}(t_1) = \varphi_n(t_1, 0)$ and $\varphi_{2n}(t_2) = \varphi_n(0, t_2)$. Consequently we have

$$\lim_{n\to\infty} \varphi_{1n}(t_1) = \varphi(t_1, 0), \quad \lim_{n\to\infty} \varphi_{2n}(t_2) = \varphi(0, t_2), \quad |t_i| < \alpha_i, \quad (i = 1, 2).$$

Therefore (a) follows immediately from Theorem 2.

To prove (b), we may notice that according to the Helly principle of selection applied to the sequence $\{F_n(x_1, x_2)\}$, there exists a subsequence $\{F_{n_k}(x_1, x_2)\}$, selected from the sequence $\{F_n(x_1, x_2)\}$ which is convergent to some function $F(x_1, x_2)$ continuous on the right and with non-negative second difference. But $F(x_1, x_2)$ must be a d.f. according to the Theorem 5, since the relation (6) is satisfied by the sequence $\{F_{n_k}(x_1, x_2)\}$. Moreover, the limit of the sequence $\{\varphi_{n_k}(t_1, t_2)\}$, namely $\varphi(t_1, t_2)$, when considered in a sufficiently small neighborhood of the origin, is the m.g.f. of $F(x_1, x_2)$. Since the d.f. is uniquely determined by its m.g.f., it follows that every convergent subsequence of $\{F_n(x_1, x_2)\}$ converges to the same limit $F(x_1, x_2)$ at each point of continuity of $F(x_1, x_2)$. This is, however, the same as to say that the sequence $\{F_n(x_1, x_2)\}$ itself converges to $F(x_1, x_2)$ at each point of continuity of $F(x_1, x_2)$.

To prove (c), we have to show that the m.g.f. of $F(x_1, x_2)$, say $\varphi^*(t_1, t_2)$, exists for $|t_i| < \alpha_i$ and is equal to $\varphi(t_1, t_2)$, $|t_i| < \alpha_i$, $(i = 1, 2)$. (We have proved that $\varphi^*(t_1, t_2) = \varphi(t_1, t_2)$ only for sufficiently small values of $|t_1|$ and $|t_2|$). The existence of $\varphi^*(t_1, t_2)$ for $|t_i| < \alpha_i$, $(i = 1, 2)$, can be easily established by the method used by Curtiss [1, p. 433]. Suppose indeed that $\varphi^*(t_1, t_2)$ does not exist at some point (t_1^0, t_2^0), where $|t_i^0| < \alpha_i$, $(i = 1, 2)$. That means that we can find a positive number N such that

$$\int_{-N}^{N} \int_{-N}^{N} e^{t_1^0 x_1 + t_2^0 x_2}\, dF x_1, x_2) > \varphi(t_1^0, t_2^0) \tag{17}$$

Since $\lim_{n\to\infty} F_n(x_1, x_2) = F(x_1, x_2)$ at all points of continuity of $F(x_1, x_2)$, and since

N can be so chosen that the marginal d.f.'s $F_1(x_1)$ and $F_2(x_2)$ are continuous for $x_1 = x_2 = \pm N$, it follows that

$$(18) \qquad \lim_{n\to\infty} \int_{-N}^{N} \int_{-N}^{N} e^{t_1^0 x_1 + t_2^0 x_2}\, dF_n(x_1, x_2) = \int_{-N}^{N} \int_{-N}^{N} e^{t_1^0 x_1 + t_2^0 x_2}\, dF(x_1, x_2).$$

The formulas (17) and (18) give $\lim_{n\to\infty} \varphi_n(t_1^0, t_2^0) > \varphi(t_1^0, t_2^0)$, which is impossible because $\lim_{n\to\infty} \varphi_n(t_1, t_2) = \varphi(t_1, t_2)$ for $|t_i| < \alpha_i$, $(i = 1, 2)$.

To prove that $\varphi(t_1, t_2) = \varphi^*(t_1, t_2)$ for $|t_i| < \alpha_i$, $(i = 1, 2)$, let (t_1, t_2) denote a fixed point such that $|t_i| < \alpha_i$, $(i = 1, 2)$. Clearly, $\varphi_n(tt_1, tt_2)$, $(n = 1, 2, \cdots)$, and $\varphi^*(tt_1, tt_2)$, considered as functions of the variable t, are m.g.f.'s provided that $|tt_i| < \alpha_i$, $(i = 1, 2)$. (See first part of proof of Theorem 4). Now, according to Theorem 2, the limit of the sequence $\{\varphi_n(tt_1, tt_2)\}$, namely $\varphi(tt_1, tt_2)$, $|tt_i| < \alpha_i$, $(i = 1, 2)$, is also a m.g.f. Since $\varphi(tt_1, tt_2) = \varphi^*(tt_1, tt_2)$ in a sufficiently small interval containing the point $t = 0$, it follows from the uniqueness property of the m.g.f. in the univariate case that $\varphi(tt_1, tt_2) = \varphi^*(tt_1, tt_2)$ identically for $|tt_i| < \alpha_i$, $(i = 1, 2)$. Putting $t = 1$, we find $\varphi(t_1, t_2) = \varphi^*(t_1, t_2)$, $|t_i| < \alpha_i$, $(i = 1, 2)$. Thus (c) is completely proved.

To prove (d), it is sufficient to notice that the sequence $\{\varphi_n(t_1, t_2)\}$ is uniformly continuous in each two-dimensional interval $|t_i| \leq \beta_i < \alpha_i$, $(i = 1, 2)$, (that is, for any $\epsilon > 0$, there exists a positive number $\delta = \delta(\epsilon)$ such that

$$|\varphi_n(t_1', t_2') - \varphi_n(t_1'', t_2'')| < \epsilon$$

if

$$|t_i' - t_i''| < \delta, \ |t_i'| \leq \beta_i, \quad |t_i''| \leq \beta_i, \ (i = 1, 2), \qquad (n = 1, 2, \cdots)).$$

Consequently, the sequence $\{\varphi_n(t_1, t_2)\}$ which is convergent for $|t_i| \leq \beta_i$, must be uniformly convergent if $|t_i| \leq \beta_i$, $(i = 1, 2)$.

References

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