ABSTRACTS OF PAPERS

Presented on January 25, 1947, at the Atlantic City meeting of the Institute

1. A Test of Significance of the Coefficient of Rank Correlation for more than Thirty Ranked Items. Nilan Norris, Hunter College.

Hotelling and Pabst (*Annals of Math. Stat.*, Vol. 7 (1936), p. 37) have suggested the use of the Tchebycheff inequality as an approximation for testing the significance of the coefficient of rank correlation in cases where the number of ranked items is too large to enable exact probabilities to be computed directly. A table prepared in accordance with this suggestion indicates that for values of the coefficient of rank correlation larger than .50 there is a wide range of corresponding numbers of ranked items greater than thirty for which at least the five per cent level of significance is satisfied.

For certain types of applications the conservativeness of the Tchebycheff test may be a virtue rather than a limitation.

2. A Generalized T Measure of Multivariate Dispersion. Harold Hotelling, University of North Carolina.

The problem of combining errors in two or more dimensions to measure the accuracy of firing and bombing is similar to problems occurring in industrial quality control where different measures of quality are applied to the same article, and to problems in mental testing and other fields. If the covariances were known a priori, the solution optimum in certain senses, for a multivariate normal distribution, would be the use of $\chi^2 = \Sigma\Sigma\lambda_{ij}x_ix_j$, where $[\lambda_{ij}]^{-1}$ is the covariance matrix and x_i is the deviation in the ith dimension. Since the covariances must in all known practical cases be estimated from a preliminary sample with (say) n degrees of freedom, χ^2 may be replaced by $T^2 = \Sigma\Sigma l_{ij}x_ix_j$, where $[l_{ij}]^{-1}$ is the estimated covariance matrix. This is the same T introduced by the author in 1931 as a generalization of the Student ratio t, and has the same distribution. Upon adding together the values of T^2 for different cases (e.g. for different bombs dropped with the same bombsight), a combined measure T_0^2 of over-all excellence (e.g. of the bombsight), is obtained. T_0^2 like χ^2, can be broken down into components meaningful with respect to the causal system, specifically in relation to possible sources of excessive discrepancy. Thus, if $\bar{x}_i$ is the ih coordinate of the centroid, or mean point of impact, of m bombs, we may write $T_M^2 = \Sigma\Sigma l_{ij}\bar{x}_i\bar{x}_j$, $T_D^2 = T_0^2 - T_M^2$. Then T_D is a function only of deviations from the mean point of impact. Asymptotically (for large n), T_0, T_M and T_D have the χ distribution with m, 2 and $m - 2$ degrees of freedom respectively. But the untrustworthiness of the χ distribution as an approximation is evident even with n as large as 256, for which case calculations have been made. The exact distributions of T_0 and T_D are ascertained when the number of variates p is 2, and the probability integrals are expressed as linear functions of two incomplete beta functions. In fact, T_0^2/M equals the sum of the roots of a determinantal equation of the form $| A - \lambda\beta | = 0$, where A and B are sample covariance matrices with n and m degrees of freedom respectively, and a similar relation holds for T_D^2 with m replaced by $m - 2$. T_0 and T_M have the distribution published in 1931, with probability integral expressible in terms of a single incomplete beta function or the variance ratio distribution. It is shown that such parameters as the circular mean deviation are best estimated with the help of the T measures, not directly by averaging individual circular deviations.

3. Asymptotic Properties of Maximum and Quasi-Maximum Likelihood Estimates. Herman Rubin, Cowles Commission for Research in Economics.

The results of J. L. Doob (*Trans. Am. Math. Soc.*, Vol. 36 (1934), pp. 759–775) on consistency of maximum likelihood estimates, are generalized and extended to arbitrary measure spaces. In some special cases, results on asymptotic normality of maximum likelihood estimates can be generalized to quasi-maximum likelihood estimates (estimates based on the assumption of a likelihood function which need not be the true function).

4. The Asymptotic Distribution of the Range. E. J. GUMBEL, Newark College of Engineering.

The asymptotic distribution of the range w for initial unlimited distributions of the exponential type is obtained by convolution of the asymptotic distributions of the two extremes. Let α and u be the parameters of the distributions of the extremes for a symmetrical variate, and let $R = \alpha(w - 2u)$ be the reduced range. Then the probability $\Psi(R)$ of the reduced range is subject to the differential equation $\Psi'' + \Psi' - \Psi \exp(-R) = 0$ which may be transformed into Bessel's equation of the first order by the substitutions $R = 2(\log 2 - \log z)$, and $\Psi = zU$. The solution is $\Psi(R) = zK_1(z)$ for the asymptotic probability, and $\psi(R) = (z^2/2)K_0(z)$ for the asymptotic distribution, $K_0(z)$ and $K_1(z)$ being the modified Bessel function of the second kind of orders zero and unity. Thus tables of $\Psi(R)$ and $\psi(R)$ may be calculated for any symmetrical distribuion of the exponential type. The distribution of the range w for normal samples of size 10 is already very close to the asymptotic distribution provided that the parameters α and u are determined from the mean and the standard deviation of the range. This method permits the calculation of the distribution of the range for normal samples of any size larger than 10.

5. The Corner Test for Association. JOHN W. TUKEY, Princeton University, and PAUL S. OLMSTEAD, Bell Telephone Laboratories.

Construction. In a scatter diagram, draw the two medians, that is, the median of the x values without regard to the values of y, and the median of the y values without regard to the values of x. Think of the four quadrants thus formed as being labelled $+, -, +, -$ in order, so that the two positive quadrants lie along one diagonal and the two negative along the other. Beginning at the right-hand side of the diagram, count in along the observations until forced to cross the horizontal median. Write down the number of observations met before this crossing, attaching the sign, $+$, if they lay in the $+$ quadrant, and the sign, $-$, if they lay in the $-$ quadrant. Repeat this process, moving up from below, moving to the right from the left, and moving down from above. The quantity to be used in the test is the algebraic sum of the four numbers thus written down.

Distribution. The exact distribution of this quantity when no association is present and no two x's and no two y's are alike is almost independent of sample size over the range of values where it is apt to be used. For example, a sum of 9 or more is expected less than one time in ten for all samples of size 6 or more; a sum of 15 or more, less than one time in 100 for all samples of size 10 or more; and a sum of 21 or more, less than one time in 1000 for all samples of size 14 or more. Even for infinite sample size, the sums for these fractions become only 9, 14, and 19, respectively.

Extensions. The same ideas that underlie the outside corner test for two variables may be extended in several ways to give tests for various types of association among three or more variables.

6. Consistent Estimates Based on Partially Consistent Observations, with Particular Reference to Structural Relations. J. NEYMAN AND ELIZABETH L. SCOTT, University of California.

Let $\{X_n\}$ be a sequence of independent random variables and let F_i denote the distribution of X_i. Each distribution F_i is assumed to depend on unknown parameters. If a parameter θ appears in an infinity of distributions F_i, it is called *structural*. Otherwise, it is *incidental*. The sequence $\{X_n\}$ is called *consistent* if $\{F_n\}$ has no incidental parameters. $\{X_n\}$ is called *partially consistent* if $\{F_n\}$ has both structural and incidental parameters.—Problem of fitting a straight line when both variables are subject to errors is that of a partially consistent series of observations. Let ξ and $\eta = \alpha + \beta\xi$ be two linearly connected quantities, perhaps related to particular stars, where α and β are unknown. The values ξ_i and η_i corresponding to the ith star, $(i - 1, 2, \cdots, s)$, are unknown. The observations provide measurements x_{ij} of ξ_i, $(j - 1, 2, \cdots, m_i)$, and measurements y_{ik}, $(k = 1, 2, \cdots, n_i)$, of η_i. Both m_i and n_i are bounded and small. On the other hand, s may be considered as increasing without limit.—Assume that the x_{ij} and the y_{ik} are normally distributed with variances σ_1^2 and σ_2^2 and means ξ_i and η_i respectively. Then the totality of observations will form a partially consistent system with the structural parameters α, β, σ_1 and σ_2 and with ξ_i as incidental parameters.—If the observable random variables are only partially consistent, then the maximum likelihood estimates of the structural parameters (a) need not be consistent, (b) even if they are consistent and asymptotically normal, alternative estimates may exist which have the same properties but smaller asymptotic variances.—Consistent estimates of structural parameters may be obtained from "modified" equations of maximum likelihood. The lower bound of the variance of estimates of structural parameters, provided by the Cramér-Rao inequality, is attained only on certain conditions which are both necessary and sufficient.