Let N be defined as above. We note that $N < \infty$ since by hypothesis $E(n) < \infty$. Let V_N be the estimate of θ when the sequential test terminates with $n = N$. Then $V_N = W/N$. Substituting this value in (16) we get

$$\frac{W}{N} - \theta = \frac{N}{E(n)} \left[\frac{W}{N} - \theta \right]. \tag{17}$$

We exclude the trivial case where $W \equiv N\theta$. Then (16) yields $E(n) = N$. That is $P(n = N) = 1$. This proves the theorem.

We remark that N may be a function of θ but for a fixed θ, $n = N$ is fixed when $\rho = 1$.

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AN EXTENSION TO TWO POPULATIONS OF AN ANALOGUE OF STUDENT'S t-TEST USING THE SAMPLE RANGE

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1. Summary. The modified t-test considered by Daly[1] (see [1]) is used to develop one-sided significance tests to decide whether the mean of a new normal population exceeds the mean of an old normal population having the same variance. Significance tests are also developed to decide whether the mean of the new population is less than the mean of the old population. These tests require very little computation for their application and are approximately as powerful as the most powerful tests of these hypotheses.

2. Introduction. Let $r_1, \cdots, r_n$, $(n \leq 10)$, be independently distributed according to a normal distribution with zero mean and unit variance. Let $r_{(u)}$ denote the uth largest of the r's. Then Daly has shown how to determine numbers g_α such that

$$\begin{aligned} &Pr[\bar{r}/(r_{(n)} - r_{(1)}) > g_\alpha] = \alpha \\ &Pr[\bar{r}/(r_{(n)} - r_{(1)}) < -g_\alpha] = \alpha. \end{aligned} \tag{1}$$

This note will use these relations to develop easily applied significance tests to decide whether the mean ν of a new normal population exceeds the mean μ of

[1] This problem is also considered by Lord in [2]. This note was in proof when [2] appeared.

an old normal population with the same variance. Significance tests are also developed to test $\nu < \mu$. The simplest case considered is that of testing a new sample value x on the basis of n past sample values $y_1, \cdots, y_n$. Then the significance test at significance level α to decide whether ν exceeds μ consists in accepting $\nu > \mu$ if

$$x > \bar{y} + g_\alpha \sqrt{n+1}[y_{(n)} - y_{(1)}],$$

where $y_{(u)}$ is the uth largest of $y_1, \cdots, y_n$.

The significance test of $\nu < \mu$ consists in accepting $\nu < \mu$ if

$$x < \bar{y} - g_\alpha \sqrt{n+1}\,[y_{(n)} - y_{(1)}].$$

These tests are generalized to the case in which x is the mean of a sample of size r from the new population, each of $y_1, \cdots, y_n$ is the mean of a sample of size s from the old population, and z is the mean of a sample of size t from the old population. Then the tests at significance level α take the form

$$\begin{aligned} &\textit{Accept } \nu > \mu \textit{ if } x > (1 - C_1)\bar{y} + C_1 z + g_\alpha[y_{(n)} - y_{(1)}]; \\ &\textit{Accept } \nu < \mu \textit{ if } x < (1 - C_1)\bar{y} + C_1 z - g_\alpha[y_{(n)} - y_{(1)}], \end{aligned} \tag{2}$$

where C_1 is a given constant which is selected by the person applying the test. The introduction of the terms z and C_1 allows less reliable past information to be utilized by lumping it together in the z term and using the constant C_1 to weight this information according to its relative importance with respect to the y's.

The power of test (2) is compared with that of the corresponding Student t-test for the case $C_1 = 0$ and $n \leq 10$. In this comparison the quantities $x, y_1, \cdots, y_n$ are considered to be the given sample values which are used for the test, that is, the quantities from which the means $x, y_1, \cdots, y_n$ were formed are not given. It is found that the power of the Student t-test is only slightly greater than that of the corresponding test (2). For the cases considered, however, it is well known that the most powerful test of $\nu > \mu$ using the quantities $x, y_1, \cdots, y_n$ is the appropriate Student t-test. Similarly for testing $\nu < \mu$. Thus the tests (2) considered are approximately as powerful as the most powerful tests of $\nu > \mu$ and $\nu < \mu$ which use $x, y_1, \cdots, y_n$.

Examination of (2) shows that the amount of computation required for the application of one of these tests is small. Consequently the tests (2) have the desirable properties of being easily computed and nearly as powerful as any tests which could be used for the given hypotheses. This suggests their use in repetitive testing procedures which are concerned with the testing of the mean of a new sample on the basis of the means of previous samples.

3. Statement of tests. In this section three significance tests of increasing generality are stated. It is to be observed that each test is a particular example of the test following it so that tests (A) and (B) are special cases of test (C).

The reason for stating tests (A) and (B) is that these tests have a much simpler appearance and will cover most cases of practical application.

(A). Let each of x, $y_1, \cdots, y_n$ represent the mean of a sample of size r; let the values of the sample whose mean is x have the distribution $N(\nu, \sigma^2)$ and the values of the samples whose means are $y_1, \cdots, y_n$ have distribution $N(\mu, \sigma^2)$, where the notation $N(\xi, \sigma^2)$ denotes the normal distribution with mean ξ and variance σ^2. Then the significance test of $\nu > \mu$ at significance level α is

$$\textit{Accept} \quad \nu > \mu \quad \textit{if} \quad x > \bar{y} + g_\alpha \sqrt{\frac{n+1}{r}}\,[y_{(n)} - y_{(1)}].$$

The significance test to decide whether $\nu < \mu$ is

$$\textit{Accept} \quad \nu < \mu \quad \textit{if} \quad x < \bar{y} - g_{(\alpha)} \sqrt{\frac{n+1}{r}}\,[y_{(n)} - y_{(1)}].$$

(B). Let x equal the mean of r sample values from $N(\nu, \sigma^2)$ and each of $y_1, \cdots, y_n$ equal the mean of s sample values from $N(\mu, \sigma^2)$. The significance test for $\nu > \mu$ at significance level α is

$$\textit{Accept} \quad \nu > \mu \quad \textit{if} \quad x > \bar{y} + g_\alpha \sqrt{\frac{n}{r} + \frac{1}{s}}\,[y_{(n)} - y_{(1)}].$$

The test of $\nu < \mu$ is given by

$$\textit{Accept} \quad \nu < \mu \quad \textit{if} \quad x < \bar{y} - g_\alpha \sqrt{\frac{n}{r} + \frac{1}{s}}\,[y_{(n)} - y_{(1)}].$$

(C). Let x equal the mean of r sample values from $N(\nu, \sigma^2)$, each of $y_1, \cdots, y_n$ equal the mean of a sample of size s from $N(\mu, \sigma^2)$, z equal the mean of a sample of size t from $N(\mu, \sigma^2)$, and C_1 be a given constant value. Then the significance test of $\nu > \mu$ at significance level α is

Accept $\nu > \mu$ *if*

$$x > (1 - C_1)\bar{y} + C_1 z + [y_{(n)} - y_{(1)}]g_\alpha \cdot \sqrt{\left(\frac{1}{r} + \frac{C_1^2}{t}\right)\left(n + \frac{(1 - C_1)^2}{s\left(\frac{1}{r} + \frac{1}{t}C_1^2\right)}\right)}.$$

The significance test to decide whether $\nu < \mu$ is

Accept $\nu < \mu$ *if*

$$x < (1 - C_1)\bar{y} + C_1 z - [y_{(n)} - y_{(1)}]g_\alpha \cdot \sqrt{\left(\frac{1}{r} + \frac{C_1^2}{t}\right)\left(n + \frac{(1 - C_1)^2}{s\left(\frac{1}{r} + \frac{1}{t}C_1^2\right)}\right)}.$$

Values of g_α for $\alpha = .05$ are given in Table I. These values were listed by Daly in [1].[2]

[2] Values of g_α for $\alpha = .05, .025, .01, .005, .001$. and $.0005$ are listed in Table 9 of [2] for sample sizes from 2 to 20.

4. Derivation of tests. As tests (A) and (B) are particular cases of test (C), it is sufficient to derive test (C).

TABLE I

Estimated Values of $g_{.05}$

n	$g_{.05}$
3	.882
4	.526
5	.385
6	.309
7	.260
8	.227
9	.202
10	.183

Let the quantities $x', y'_1, \cdots, y'_n, z'$ be defined by

$$x' = \frac{(x-\nu)\sqrt{r}}{\sigma}, \qquad y'_i = \frac{(y_i - \mu)\sqrt{s}}{\sigma}, \qquad (i = 1, \cdots, n),$$

$$z' = \frac{(z-\mu)\sqrt{t}}{\sigma}.$$

Then $x', y'_1, \cdots, y'_n, z'$ are independently distributed according to $N(0, 1)$. Define

$$r_u = \frac{1}{K_1}\left(K_1 y'_u - \sum_1^n y'_i + K_2 x' + K_2 C z'\right), \qquad (u = 1, \cdots, n).$$

It is easily verified that

$$E(r_u) = 0, \qquad E(r_u^2) = \frac{1}{K_1^2}\,[K_1^2 + (1 + C^2)K_2^2 - 2K_1 + n]$$

$$E(r_u r_v) = \frac{1}{K_1^2}\,[(1 + C^2)K_2^2 - 2K_1 + n], \qquad (u \neq v).$$

Thus, if K_1 and K_2 satisfy the equations

$$\begin{aligned} &\left(\sqrt{\frac{r}{s}} + C\sqrt{\frac{t}{s}}\right)K_2 + K_1 - n = 0 \\ &(1 + C^2)K_2^2 - 2K_1 + n = 0, \end{aligned} \tag{3}$$

the r_u will be independent of μ when $\mu = \nu$. Also they will be independently distributed according to $N(0, 1)$.

Rewriting the r_u in terms of $x, y_1, \cdots, y_n, z$ one obtains

$$(4) \quad r_u = \frac{\sqrt{s}}{K_1\sigma}\left[K_1 y_u - \sum_1^n y_i + K_2\sqrt{\frac{r}{s}}x + K_2 C\sqrt{\frac{t}{s}}z + K_2\sqrt{\frac{r}{s}}(\mu - \nu)\right].$$

Using (3) the mean of the r_u is found to be

$$\bar{r} = \frac{K_2\sqrt{r}}{K_1\sigma}\left[x - \left(1 + C\sqrt{\frac{t}{r}}\right)\bar{y} + C\sqrt{\frac{t}{r}}z - (\nu - \mu)\right].$$

Let $r_{(u)}$ denote the uth largest of $r_1, \cdots, r_n$. Then from (1)

$$\alpha = Pr[\bar{r}/(r_{(n)} - r_{(1)}) > g_\alpha] = Pr\left[\frac{K_2\sqrt{r}}{K_1}\left(x - \left(1 + C\sqrt{\frac{t}{r}}\right)\bar{y} + C\sqrt{\frac{t}{r}}z - (\nu - \mu)\right)\Big/ (y_{(n)} - y_{(1)}) > g_\alpha\right]$$

It is easily proved from (3) that

$$\frac{K_1}{K_2\sqrt{r}} = \pm\sqrt{\frac{1 + C^2}{r}\left(n + \frac{(\sqrt{r} + C\sqrt{t})^2}{s(1 + C^2)}\right)}.$$

Choosing the positive sign, putting $C = -\sqrt{\frac{r}{t}}\,C_1$, and letting $\mu = \nu$ one obtains

$$Pr\left[x > (1 - C_1)\bar{y} + C_1 z + [y_{(n)} - y_{(1)}]g_\alpha \cdot \sqrt{\left(\frac{1}{r} + \frac{C_1^2}{t}\right)\left(n + \frac{(1 - C_1)^2}{s\left(\frac{1}{r} + \frac{1}{t}C_1^2\right)}\right)}\,\right] = \alpha,$$

verifying the first part of test (C). The second part of test (C) is verified by choosing the negative sign for $\frac{K_1}{K_2\sqrt{r}}$ (or by repeating the above argument using the second part of (1)).

5. Power comparison with t-test. Let $x, y_1, \cdots, y_n$ satisfy the conditions of test (B) in section 3. Then Student's t using $x, y_1, \cdots, y_n$ is given by

$$t = \frac{[x - \bar{y} - (\nu - \mu)]}{\sqrt{\sum_1^n (y_i - \bar{y})^2}} \cdot \sqrt{\frac{n - 1}{s\left(\frac{1}{r} + \frac{1}{ns}\right)}}.$$

The Student t-test based on this value of t furnishes the most powerful test of $\nu > \mu$ (and $\nu < \mu$) using $x, y_1, \cdots, y_n$. The purpose of this section is to show that test (B) has approximately the same power as this Student t-test for $n \leq 10$.

Daly has shown (see [1]) that if $r_1, \cdots, r_n$ are independently distributed according to $N(\xi, \sigma^2)$, then the test based on

$$(\bar{r} - \xi)/(r_{(n)} - r_{(1)})$$

has approximately the same power for testing $\xi > 0$ (and $\xi < 0$) as the corresponding Student t-test based on

$$t = \frac{(\bar{r} - \xi)\sqrt{n(n-1)}}{\sqrt{\sum_1^n (r_i - \bar{r})^2}} \tag{5}$$

for $n \leq 10$.

Using the notation of section 4 let

$$r_u = \frac{\sqrt{s}}{K_1}\left[K_1 y_u - \sum_1^n y_i + K_2 \sqrt{\frac{r}{s}}\, x\right], \qquad (u = 1, \cdots, n),$$

where $\frac{K_1}{K_2} > 0$. Then from consideration of (4) with $C = 0$ it is seen that the r_u are independently distributed according to $N(\xi, \sigma^2)$, where ξ equals a positive constant times $(\nu - \mu)$. Following the derivations in section 4 with $C = 0$, it is seen that the test of $\xi > 0$ with this particular choice of the r_u is identical with the test of $\nu > \mu$ given in (B) of section 3. Similarly the test of $\xi < 0$ is identical with the test (B) of $\nu < \mu$. Thus the test (B) has approximately the same power for testing $\nu > \mu$ (and $\nu < \mu$) as the Student t-test based on the value of t given in (5) if $n \geq 10$. Replacing the r_u in (5) by their values in terms of $x, y_1, \cdots, y_n, n, r,$ and s, it is found that (5) becomes

$$t = \frac{[x - \bar{y} - (\nu - \mu)]}{\sqrt{\sum_1^n (y_i - \bar{y})^2}} \cdot \sqrt{\frac{n-1}{s\left(\frac{1}{r} + \frac{1}{ns}\right)}}.$$

This proves that test (B) is approximately as powerful for testing $\nu > \mu$ and $\nu < \mu$ as the most powerful test based on the quantities $x, y_1, \cdots, y_n$ if $n \leq 10$. As test (A) is a particular case of test (B), these results also apply to test (A).

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ON THE NORM OF A MATRIX

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In studying the convergence of iterative procedures in matrix computation and in setting limits of error after a finite number of steps, Hotelling [1] used the square root of the sum of squares of the elements of a matrix as its norm. A wide class of functions exists which may be employed as norms in matrix calculation and substituted directly in the expressions derived by Hotelling. The