THE PROBABILITY FUNCTION OF THE PRODUCT OF TWO NORMALLY DISTRIBUTED VARIABLES[1]

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1. Introduction and summary. Let x and y follow a normal bivariate probability function with means $\bar{X}$, $\bar{Y}$, standard deviations σ_1, σ_2, respectively, r the coefficient of correlation, and $\rho_1 = \bar{X}/\sigma_1$, $\rho_2 = \bar{Y}/\sigma_2$. Professor C. C. Craig [1] has found the probability function of $z = xy/\sigma_1\sigma_2$ in closed form as the difference of two integrals. For purposes of numerical computation he has expanded this result in an infinite series involving powers of z, ρ_1, ρ_2, and Bessel functions of a certain type; in addition, he has determined the moments, semin-variants, and the moment generating function of z. However for ρ_1 and ρ_2 large, as Craig points out, the series expansion converges very slowly. Even for ρ_1 and ρ_2 as small as 2, the expansion is unwieldy. We shall show that as ρ_1 and $\rho_2 \to \infty$, the probability function of z approaches a normal curve and in case $r = 0$ the Type III function and the Gram-Charlier Type A series are excellent approximations to the z distribution in the proper region. Numerical integration provides a substitute for the infinite series wherever the exact values of the probability function of z are needed. Some extensions of the main theorem are given in section 5 and a practical problem involving the probability function of z is solved.

2. Theorems on approach to normality. The moment generating function of z, $M_z(\theta)$, is [1]

$$M_z(\theta) = \frac{\exp \dfrac{(\rho_1^2 + \rho_2^2 - 2r\rho_1\rho_2)\theta^2 + 2\rho_1\rho_2\theta}{2[1-(1+r)\theta][1+(1-r)\theta]}}{\sqrt{[1-(1+r)\theta][1+(1-r)\theta]}}. \tag{2.1}$$

Let $\bar{z}$, and σ_z be the mean and the standard deviation of z, and $t_z = (z - \bar{z})/\sigma_z$. Now

$$\bar{z} = \rho_1\rho_2 + r, \qquad \sigma_z = \sqrt{\rho_1^2 + \rho_2^2 + 2r\rho_1\rho_2 + 1 + r^2}. \tag{2.2}$$

Using (2.2) we find in the usual way the moment generating function of t_z

$$M_{t_z} = \frac{\exp \dfrac{-2rw + (\rho_1^2 + \rho_2^2 + 2r\rho_1\rho_2)w^2 + 4r^2w^2 - 2w^3(r^2-1)(\rho_1\rho_2 + r)}{2[1-(1+r)w][1+(1-r)w]}}{\sqrt{[1-(1+r)w][1+(1-r)w]}}, \tag{2.3}$$

where $w = \theta/\sigma_z$.

[1] Presented to the American Mathematical Society, Oct. 28, 1944, New York City.

Consider $r \geqq 0$. Then in the limit as ρ_1 and $\rho_2 \rightarrow \infty$ in any manner whatever,

$$\lim_{\rho_1, \rho_2 \rightarrow \infty} M_{t_z}(\theta) = e^{\theta^2/2}, \tag{2.4}$$

and by the theorem of Curtiss [2] on moment generating functions we see in the limit as ρ_1, $\rho_2 \rightarrow \infty$ the probability function of z approaches a normal curve with mean, $\bar{z}$, and variance σ_z^2, $r \geqq 0$.

In case $-1 + \epsilon < r < 0$, $\epsilon > 0$, some care is required wherever

$$\sqrt{\rho_1^2 + \rho_2^2 + 2\rho_1 \rho_2 r}$$

occurs. If one uses $\rho_1^2 + \rho_2^2 \geqq 2\rho_1\rho_2$, the proof goes forward quite readily. Hence we have proved the theorem:

THEOREM (2.5). *The distribution of z approaches normality with mean $\bar{z}$, and variance σ_z^2 as ρ_1 and $\rho_2 \rightarrow \infty$ in any manner whatever, $-1 + \epsilon < r \leqq 1$, $\epsilon > 0$.*

It is evident in Theorem (2.5) we may allow ρ_1, $\rho_2 \rightarrow -\infty$ without any other changes. Theorems (2.6) and (2.7) are proved in essentially the same way as (2.5).

THEOREM (2.6). *The distribution of z approaches normality with mean $\bar{z}$, and variance σ_z^2, if $\rho_1 \rightarrow \infty$, $\rho_2 \rightarrow -\infty$, $-1 \leqq r < 1 - \epsilon$, $\epsilon > 0$.*

THEOREM (2.7). *The distribution of z approaches normality if ρ_1 remains constant $\rho_2 \rightarrow \infty$, $-1 + \epsilon < r \leqq 1$, $\epsilon > 0$; or if ρ_1 remains constant $\rho_2 \rightarrow -\infty$, $-1 \leqq r < 1 - \epsilon$, $\epsilon > 0$.*

Naturally in any of the theorems ρ_1 and ρ_2 may be interchanged. In practice ρ_1 and ρ_2 are usually positive. The approach to normality is more rapid if both ρ_1 and ρ_2 have the same sign as r.

3. Numerical values. In order to show how closely the Type III and the Gram-Charlier Type A series approximate the probability function of z, $f(z)$, or more precisely $f(z, \rho_1, \rho_2, r)$, we use numerical integration where

$$f(z, \rho_1, \rho_2, r) = I_1(z) - I_2(z),$$

$$I_1(z) = \frac{1}{2\pi\sqrt{1 - r^2}} \int_0^\infty \exp - \frac{1}{2(1 - r^2)} \left\{(x - \rho_1)^2 - 2r(x - \rho_1)\left(\frac{z}{x} - \rho_2\right) + \left(\frac{z}{x} - \rho_2\right)^2\right\} \frac{dx}{x}, \tag{3.1}$$

and $I_2(z)$ is the integral of the same function over $(-\infty, 0)$, [1]. Now $I_1(z)$ may be written as

$$I_1(z) = \frac{1}{\sqrt{1 - r^2}} \int_0^\infty \varphi(t_1)\varphi(t_2)\beta(t_3) \frac{dx}{x}, \tag{3.2}$$

where

$$\varphi(t) = \frac{e^{-(t^2/2)}}{\sqrt{2\pi}}, \qquad t_1 = \frac{x - \rho_1}{\sqrt{1 - r^2}}, \qquad t_2 = \left(\rho_2 - \frac{z}{x}\right) \Big/ \sqrt{1 - r^2},$$

$$\beta(t_3) = e^{t_3}, \; t_3 = rt_1t_2.$$

We readily obtain $I_1(z)\sqrt{1-r^2}$ by forming the product of $\varphi(t_1)$, $\varphi(t_2)$, $\beta(t_3)$, and $1/x$ using numerical integration applying Weddle's formula, the Gregory-Newton formula, or the simple rectangular formula depending on circumstances. The rectangular formula [3] is remarkably accurate when the function $T = \varphi(t_1)\varphi(t_2)\beta(t_3)/x$ in the interval 0 to ∞ or 0 to $-\infty$ is somewhat symmetrical. Appropriate tables for $\varphi(t_1)$, $\varphi(t_2)$ (see [4]), $\beta(t_3)$ (see [5]) and $1/x$ (see [6]) are readily available. In the important case of the independence of x and y, $r = 0$ and (3.2) becomes

$$(3.3) \qquad I_1(z) = \int_0^\infty \varphi(t_1)\varphi(t_2)\frac{dx}{x}, \qquad t_1 = x - \rho_1, \qquad t_2 = \rho_2 - \frac{z}{x}.$$

4. Approximations to $f(z)$. When $r = 0$, the standard seminvariants ξ_3, and ξ_4 of z are

$$(4.1) \qquad \xi_3 = \frac{6\rho_1\rho_2}{(\rho_1^2+\rho_2^2+1)^{3/2}}, \qquad \xi_4 = \frac{6\{2(\rho_1^2+\rho_2^2)+1\}}{(\rho_1^2+\rho_2^2+1)^2}$$

remembering

$$\bar{z} = \rho_1\rho_2, \; \sigma_z = \sqrt{\rho_1^2+\rho_2^2+1}.$$

In the Pearson system (see [7]) δ, the criterion, is

$$(4.2) \qquad \delta = \frac{2\xi_4 - 3\xi_3^2}{6+\xi_4}$$

and for the probability function of z

$$(4.3) \qquad \delta = \frac{2(\rho_1^2+\rho_2^2+1)\{2(\rho_1^2+\rho_2^2)+1\} - 18\rho_1^2\rho_2^2}{(\rho_1^2+\rho_2^2+1)[(\rho_1^2+\rho_2^2+1)^2+2(\rho_1^2+\rho_2^2)+1]}$$

and if $\rho_1 = \rho_2 = \rho$

$$(4.4) \qquad \delta = \frac{2(4\rho^2+1)(2\rho^2+1) - 18\rho^4}{(2\rho^2+1)[(2\rho^2+1)^2+(4\rho^2+1)]}.$$

Now $\delta = 0$, $\xi_3 \neq 0$, for the Type III function, and clearly $\lim_{\rho_1,\rho_2\to\infty} \delta = 0$. By use of (3.3) the accurate values of $f(z)$ have been calculated for various combinations of ρ_1 and ρ_2 and compared with the Type III approximation using $\bar{z}$, σ_z, ξ_3.

(4.5) Investigations so far completed show that for $\rho_1 \geqq 4$ and $\rho_2 \geqq 4$ simultaneously, and $|\delta| \leqq .008$, the Type III approximation will provide values of t_z correct to three significant figures at least where

$$(4.6) \qquad \int_{-\infty}^{t_z^{(1)}} f(t_z) = \alpha, \qquad \int_{t_z^{(2)}}^{\infty} f(t_z) = \alpha, \quad \text{and} \quad .05 \leqq \alpha \leqq .005.$$

These are the values of t_z which would be needed in testing hypotheses. The exact values of $t_z^{(1)}$ and for $t_z^{(2)}$ for various values of ρ_1 and ρ_2 less than 4 will be determined it is hoped in the future and will be published along with the comparisons of the Type III values of t_z with the accurate values of t_z in the important borderline cases of $\rho_1 = \rho_2 = 2$, and $\rho_1 = \rho_2 = 3$. The values of $f(z)$ for $\rho_1 = \rho_2 = 2$ and $\rho_1 = \rho_2 = 4$ have been calculated but these are being withheld for a more complete table. The table of values of $\bar{z}$, σ_z, ξ_3, ξ_4, and δ (Table II) shows then that the Type III function is excellent along a band about $\rho_1 = \rho_2$, since $\xi_3 \neq 0$, and δ is very small.

We use the Gram-Charlier Type A series of three terms to approximate the probability function of z in t_z units.

$$f(t_z) \sim \varphi(t) - \frac{\xi_{3:z}}{3!}\varphi^{(3)}(t) + \frac{\xi_{4:z}}{4!}\varphi^{(4)}(t), \tag{4.7}$$

in the usual notation.

TABLE I

t_z	$f(t_z)$ Correct value	Normal Curve	Gram-Charlier Type A
.9950372	.2406367	.2431716	.2408235
1.4925558	.1275209	.130970	.127484
1.9900744	.0538243	.0550708	.053704
2.4875930	.0184606	.0180791	.0184500
2.9851116	.0052477	.0046338	.0052944
3.4826302	.0012609	.0009272	.0012804
3.9801488	.0002611	.0001449	.000260
4.4776674	.0000467	.0000177	.0000425
4.9751860	.00000745	.00000168	.00000555

(4.8) For $|\xi_3| < .5$ and $\xi_4 < .4$ simultaneously the Gram-Charlier Type A series is quite adequate for finding probability levels such as those of (4.6). These will in general give 3 significant figures for $t_z^{(1)}$ or $t_z^{(2)}$. In the special case $\rho_1 = 0$, $\rho_2 = 10$, the Gram-Charlier Type A series differs from $f(t_z)$ very slightly in the range $1 \leqq |t_z| < \infty$ (see Table 1). Naturally the Gram-Charlier will be used wherever Type III is not indicated, although there exist some overlapping regions where either one may be used. It should be noticed that the approach of $f(z)$ to normality is more rapid along a row than down a diagonal. In case either ρ_1 or ρ_2 is negative, we may make use of the equation

$$f(z, -\rho_1, \rho_2, r) = f(-z, \rho_1, \rho_2, -r). \tag{4.9}$$

We note that when $r = 0$, $f(z, \rho_1, \rho_2)$ always possesses a discontinuity at $z = 0$, (see [1]). A table of $\bar{z}$, σ_z, ξ_3, ξ_4, and δ is provided for values of ρ_1 and ρ_2 from 0 to 10 inclusive.

TABLE II*

ρ_2 \ ρ_1	2	4	6	8	10
0	0 2.236068 0 2.160 .529	0 4.123106 0 .685121 .205	0 6.082762 0 .319942 .101	0 8.062258 0 .183195 .059	0 10.049876 0 .118224 .039
2	4 3 .8 1.259259 .020	8 4.582576 .498784 .557823 .056	12. 6.403124 .274256 .289114 .056	16. 8.306624 .167493 .172653 .042	20. 10.246951 .111531 .113742 .031
4		16. 5.744563 .506408 .358127 −.0084	24. 7.280110 .373206 .224279 .0049	32. 9. .263374 .147234 .014	40 10.816654 .189641 .102126 .016
6			36. 8.544004 .346314 .163258 −.0054	48. 10.049876 .28373 .118224 −.00083	60 11.704700 .224503 .087272 .0038
8				64. 11.357817 .262088 .092663 −.0034	80 12.845233 .226472 .072507 −.0015
10					100. 14.177447 .210551 .059553 −.0023

* The first value in a cell is $\bar{z}$, the second σ_3, the third ξ_3, the fourth ξ_4, the fifth δ.

5. Some extensions. We may generalize our results to any case where x and y are distributed approximately in a normal distribution such as the distribution of the product of two means, when the sizes of the samples N_1 and N_2 are large and consequently ρ_1 and ρ_2 will be large. Another example occurs if x and y each follows a Bernoulloi probability function with parameters p_1 and p_2 respectively where the number of trials in each case is large. We must warn the reader that the condition $\rho_1 \to \infty$, $\rho_2 \to \infty$ alone does not mean that the distribution of z approaches normality. Both x and y must be distributed normally.

The actual problem which gave rise to this investigation was the question of determining the sum of a great many variates [8]. Let T variates $v_1, v_2, \cdots, v_T$ be given whose sum $A = \sum_{i=1}^{T} v_i$ is desired. Clearly

$$A = T\bar{V}_p, \bar{V}_p = \sum_{i=1}^{T} v_i/T.$$

Now let us estimate A by $\tilde{A} = \tilde{T}_s\bar{V}_s$ where $\tilde{T}_s$ is an estimate of T and $\bar{V}_s$ is an estimate of $\bar{V}_p$. If $\sigma_{\tilde{T}_s}$ is very small, $\rho_1 = T/\sigma_{\tilde{T}_s}$ will be large and $\rho_2 = \bar{V}_p/\sigma_{\bar{V}_s} = \sqrt{N}\bar{V}_p/\sigma_p$ will be very large. Assuming $\tilde{T}_s$ is distributed normally and obviously $\bar{V}_s$ is distributed normally for N large, we see by the theorems of this paper that $\tilde{A}$ will be distributed normally. Confidence limits for A may be calculated in the usual fashion as $\tilde{A} \pm \gamma\sigma_{\tilde{A}}$, where γ is determined by

$$\int_{t=\gamma}^{\infty} \varphi(t)dt = \alpha,$$

with α generally chosen as .025 or less and

$$\sigma_{\tilde{A}} = \sqrt{\tilde{T}_s^2\sigma_{\bar{V}_s}^2 + \bar{V}_s^2\sigma_{\tilde{T}_s}^2 + \sigma_{\bar{V}_s}^2\sigma_{\tilde{T}_s}^2}.$$

Stratification is also possible. It is interesting to note that many functions which occur in life insurance are products. Such applications will be treated fully elsewhere. Naturally the critical region whether both tails or one tail of the distribution should be used depends on the alternatives to the hypothesis being tested.

Generalizations of the main theorem are possible for the probability function of $z = \prod_{i=1}^{r} x_i$ where $x_1, x_2, \cdots, x_r$ follow a multivariate normal probability function. These will be investigated in a later paper. It may be noted that J. B. S. Haldane has investigated the distribution of a product along different lines [9].

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