corresponding value of R_3 computed for a chosen p, then approximately, the proportion p' of plotted errors should fall within the circle of radius R_3.

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A NOTE ON THE EFFICIENCY OF THE WALD SEQUENTIAL TEST

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The sequential likelihood ratio test of Wald for testing the hypothesis H_0 that the probability density function is $f(X, \theta_0)$ against the one-sided alternative H_1 that the function is $f(X, \theta_1)$ has been shown [1] to have the optimum property of minimizing the expected number of observations at the two points $\theta = \theta_0$ and $\theta = \theta_1$. Tables showing the actual magnitude of the percentage saving of this sequential procedure compared with the classical "best" non-sequential test have been calculated (see [1], page 147) for the normal case when

$$f(X, \theta) = \frac{1}{\sqrt{2\pi}} \exp \frac{-(X - \theta)^2}{2}.$$

In this note we will show that when θ_1 is close to θ_0, the percentage saving is independent of the particular function $f(X, \theta)$ and the particular values θ_1 and θ_0, so that the tables mentioned above can be used to show the percentage saving for any one-sided sequential test involving a single parameter, provided $f(X, \theta)$ satisfies some weak restrictions.

Let $f(X, \theta)$ be the probability density function of a random variable. Let $E_i(n)$ denote the expected value (when $\theta = \theta_i$) of the number of independent observations required by the Wald sequential procedure to test the hypothesis H_0 that $\theta = \theta_0$ against $\theta = \theta_1 = \theta_0 + \Delta$ with probabilities α of rejecting H_0 when $\theta = \theta_0$ and β of accepting H_0 when $\theta = \theta_1$. Let N be the number of independent observations required to achieve the same probabilities α and β for testing the hypothesis $\theta = \theta_0$ against $\theta = \theta_1$ by the most powerful non-sequential test. Let U_α and U_β be defined by the relations

$$\alpha = \frac{1}{\sqrt{2\pi}} \int_{U_\alpha}^{\infty} \exp\left\{-\frac{t^2}{2}\right\} dt$$

and

$$\beta = \frac{1}{\sqrt{2\pi}} \int_{U_\beta}^{\infty} \exp\left\{-\frac{t^2}{2}\right\} dt.$$

We will prove the following theorem:

$$\underset{\Delta=\theta_1-\theta_0\to 0}{\text{Limit}} \left\{\frac{E_0(n)}{N}\right\} = -2\,\frac{\left\{\alpha \log\left(\frac{1-\beta}{\alpha}\right) + (1-\alpha)\log\left(\frac{\beta}{1-\alpha}\right)\right\}}{(U_\alpha + U_\beta)^2}$$

provided $f(X, \theta)$ *satisfies the following conditions:*

(*A*) $\int_{-\infty}^{\infty} f(X, \theta)\, dx$ *can be differentiated twice under the integral sign with respect to* θ.

(*B*) *All four of the integrals*

$$\int_{-\infty}^{\infty}\left\{\frac{f''(x,\theta^*)}{f(x,\theta^*)} - \left[\frac{f'(x,\theta^*)}{f(x,\theta^*)}\right]^2\right\} f(x,\theta_0)\,dx,$$

$$\int_{-\infty}^{\infty}\frac{f'(x,\theta_0)}{f(x,\theta_0)}\, f'(x,\theta^*)\,dx,$$

$$\int_{-\infty}^{\infty}\left[\frac{f'(x,\theta_0)}{f(x,\theta_0)}\right]^2 f(x,\theta^*)\,dx,$$

$$\int_{-\infty}^{\infty}\frac{f'(x,\theta_0)}{f(x,\theta_0)}\, f(x,\theta^*)\,dx,$$

are continuous functions of θ^* *at* $\theta^* = \theta_0$. *A sufficient condition for* (*B*) *is that all the integrals be uniformly convergent with respect to* θ^* *in some interval* $\theta_0 \le \theta^* \le \theta_0 + \Delta$, *and all the integrands be continuous functions of* X *and* θ^*. *A similar theorem holds regarding the limit of* $\underset{\Delta\to 0}{\ } \left\{\frac{E_1(n)}{N}\right\}$.

The proof is as follows: From [1], we know that

$$E_0(n) = \frac{\alpha \log\left(\frac{1-\beta}{\alpha}\right) + (1-\alpha)\log\left(\frac{\beta}{1-\alpha}\right)}{E_0(z)} + o(1),$$

where

$$z = \log\left[\frac{f(x,\theta_1)}{f(x,\theta_0)}\right]$$

and $o(1) \to 0$ as $\Delta \to 0$.

Now

$$E_0(z) = \int_{-\infty}^{\infty}\left[\log\left(\frac{f(x,\theta_1)}{f(x,\theta_0)}\right)\right] f(x,\theta_0)\,dx,$$

$$= \int_{-\infty}^{\infty}[\log f(x,\theta_0+\Delta)]\,f(x,\theta_0)\,dx - \int_{-\infty}^{\infty}[\log f(x,\theta_0)]\,f(x,\theta_0)\,dx.$$

Expanding $\log f(x, \theta_0 + \Delta)$ in a Taylor series about $\Delta = 0$, we have

$$\log f(x, \theta_0 + \Delta) = \log f(x, \theta_0) + \Delta \frac{f'(x, \theta_0)}{f(x, \theta_0)} + \frac{\Delta^2}{2}\left[\frac{ff'' - f'^2}{f^2}\right]_{\theta=\theta_0} + \frac{\Delta^2}{2} R_1,$$

where

$$\theta_0 \le \theta^* \le \theta_0 + \Delta, \qquad f' = \frac{\partial f(x, \theta)}{\partial \theta}, \qquad f'' = \frac{\partial^2 f(x, \theta)}{\partial \theta^2},$$

and

$$R_1 = \left[\frac{ff'' - f'^2}{f^2}\right]_{\theta=\theta_0}^{\theta=\theta^*}$$

From assumption (A) we find that

$$\int_{-\infty}^{\infty} f'(x, \theta_0)\, dx = 0 \text{ and } \int_{-\infty}^{\infty} f''(x, \theta_0)\, dx = 0,$$

while from assumption (B)

$$\int_{-\infty}^{\infty} R_1 f(x, \theta_0)\, dx \to 0 \text{ as } \Delta \to 0.$$

Therefore

$$E_0(z) = -\frac{\Delta^2}{2}\left[\int_{-\infty}^{\infty}\left[f\left(\frac{f'}{f}\right)^2\right]_{\theta=\theta_0} dx + o\,(1)\right].$$

To find N for the most powerful non-sequential test, we make use of the fact (see [2]) that an asymptotically most powerful test for one-sided alternatives is given by a region of the type

$$U_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{i=N} \frac{f'(x_i, \theta_0)}{f(x_i, \theta_0)} \ge K.$$

When $\Delta \to 0$, $N \to \infty$, and since U_N is the sum of N independent variates with a finite second moment, the distribution of $\frac{U_N - E(U_N)}{\sigma_{U_N}}$ approaches that of a normal variate with zero mean and unit variance. Hence we find the N required for a test with Type I and Type II errors α and β by solving for N from the relations

$$\frac{K}{\sqrt{E_0\left(\frac{f'}{f}\right)^2_{\theta=\theta_0}}} = U_\alpha \tag{1}$$

and

$$\frac{K - \sqrt{N}\, E_1\left(\frac{f'}{f}\right)_{\theta=\theta_0}}{\sqrt{E_1\left(\frac{f'}{f}\right)^2_{\theta=\theta_0} - \left[E_1\left(\frac{f'}{f}\right)_{\theta=\theta_0}\right]^2}} = -U_\beta \tag{2}$$

Now let $y = \left(\frac{f'}{f}\right)_{\theta=\theta_0}$, and we find from (1) and (2) that

$$N = \left[\frac{U_\alpha\sqrt{E_0(y^2)} + U_\beta\sqrt{E_1(y^2) - [E_1(y)]^2}}{E_1(y)}\right]^2.$$

Now

$$E_1(y) = \int_{-\infty}^{\infty}\frac{f'(x,\theta_0)}{f(x,\theta_0)}f(x,\theta_1)\,dx$$

$$= \Delta\int_{-\infty}^{\infty}\frac{f'(x,\theta_0)}{f(x,\theta_0)}f'(x,\theta_0)\,dx + \Delta\int_{-\infty}^{\infty}\frac{f'(x,\theta_0)}{f(x,\theta_0)}[f'(x,\theta)]_{\theta=\theta_0}^{\theta=\theta^*}\,dx$$

$$= \Delta E_0 y^2[1 + o(1)] \text{ from assumption } B.$$

Proceeding in a similar manner, we find

$$[U_\alpha\sqrt{E_0(y)^2} + U_\beta\sqrt{E_1(y)^2 - [E_1(y)]^2}]^2 = E_0(y^2)[U_\alpha + U_\beta(1 + o(1))]^2.$$

We now have

$$\frac{E_0(n)}{N} = \frac{\Delta^2[E_0(y^2)]^2(1 + o(1))^2}{E_0(y^2)[U_\alpha + U_\beta(1 + o(1))]^2} \times \frac{\alpha\log\left(\frac{1-\beta}{\alpha}\right) + (1-\alpha)\log\left(\frac{\beta}{1-\alpha}\right)}{-\frac{\Delta^2}{2}[E_0(y^2) + o(1)]}$$

therefore

$$\lim_{\Delta\to 0}\left\{\frac{E_0(n)}{N}\right\} = -2\frac{\left[\alpha\log\left(\frac{1-\beta}{\alpha}\right) + (1-\alpha)\log\left(\frac{\beta}{1-\alpha}\right)\right]}{(U_\alpha + U_\beta)^2}.$$

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[2] A. WALD, "Some examples of asymptotically most powerful tests", *Annals of Math. Stat.*, Vol. 12 (1941).

A NOTE ON THE POISSON-CHARLIER[1] FUNCTIONS

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The polynomials $p_n(m, z)$ given by the definition

$$(1) \qquad p_n(m, z) \equiv (-)^m e^z z^{-m}\frac{d^n}{dz^n}[e^{-z}z^m],$$

[1] This note was written while the author was employed by the Radiation Laboratory, M.I.T.