ON THE CHARLIER TYPE B SERIES

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1. Introduction. The Type B series of Charlier has been discussed in some detail in the literature (See references at the end of the paper). The problem of the convergence of the Type B series has been considered by Pollaczek-Geiringer [12], [13], Szegö [12] (page 110), Uspensky [16], Jacob [5], Schmidt [16] and Obrechkoff [11]. There is presented in the following a method of development of the Type B series which is believed to be of some interest, including a necessary and sufficient condition for the convergence which is basically the same as that of Schmidt [16]. A result of Steffensen [17] is extended and shown to be related to the Charlier Type B series.

2. Statement of results. Consider the function $p(r)$, defined for $r = 0, 1, 2, \cdots$, and such that

$$\sum_{r=0}^{\infty} p(r) = 1; \qquad \sum_{r=0}^{\infty} |p(r)| = A \tag{2.1}$$

where A is some finite value. Let the n-th factorial moment be defined by

$$\begin{aligned} \mu_{(0)} &= 1 \\ \mu_{(n)} &= \sum_{r=0}^{\infty} r(r-1)(r-2)\cdots(r-n+1)p(r), \qquad (n = 1, 2, \cdots) \end{aligned} \tag{2.2}$$

For arbitrary λ let

$$\begin{aligned} L_n = \mu_{(n)} - n\mu_{(n-1)}\lambda + \frac{n(n-1)}{2!}\mu_{(n-2)}\lambda^2 \\ - \frac{n(n-1)(n-2)}{3!}\mu_{(n-3)}\lambda^3 + \cdots + (-1)^n\lambda^n. \end{aligned} \tag{2.3}$$

We prove the following results:

Theorem. *A necessary and sufficient condition that the function* $p(r)$ *of* (2.1) *may be expressed by the absolutely convergent series*

$$p(r) = \frac{e^{-\lambda}\lambda^r}{r!} + L_1 \frac{\partial}{\partial\lambda}\frac{e^{-\lambda}\lambda^r}{r!} + \frac{L_2}{2!}\frac{\partial^2}{\partial\lambda^2}\frac{e^{-\lambda}\lambda^r}{r!} + \cdots \tag{2.4}$$

is that

$$1 + |\mu_{(1)}| + \tfrac{1}{2!}|\mu_{(2)}| + \tfrac{1}{3!}|\mu_{(3)}| + \cdots + \frac{1}{n!}|\mu_{(n)}| + \cdots \tag{2.5}$$

converges where L_n *is defined as in* (2.3).

3. Generating functions. For the function $p(r)$ of (2.1) consider the generating function defined by

$$\varphi(z) = \sum_{r=0}^{\infty} z^r p(r) \tag{3.1}$$

where z is a complex variable. Because of (2.1) it is clear that the right member of (3.1) is uniformly and absolutely convergent for $|z| \leqq 1$ so that the radius of convergence of (3.1) is some value $R_1 > 1$.

The Taylor expansion of $\varphi(z)$ about the point $z = 1$ is given by

$$\varphi(z) = \varphi(1) + (z - 1)\varphi'(1) + \frac{(z-1)^2}{2!}\varphi''(1) + \cdots \tag{3.2}$$

where, as may be readily obtained from (3.1),

$$\varphi^{(n)}(1) = \sum_{r=0}^{\infty} r(r-1)(r-2)\cdots(r-n+1)p(r) = \mu_{(n)} . \tag{3.3}$$

If it is assumed that (2.5) converges, then

$$\varphi(z) = 1 + (z-1)\mu_{(1)} + \frac{(z-1)^2}{2!}\mu_{(2)} + \cdots + \frac{(z-1)^n}{n!}\mu_{(n)} + \cdots \tag{3.4}$$

is uniformly and absolutely convergent for $|z - 1| \leqq 1$.

4. Sufficiency. For arbitrary λ let us set

$$\begin{aligned} e^{-\lambda(z-1)} &\left(1 + \mu_{(1)}(z-1) + \mu_{(2)}\frac{(z-1)^2}{2!} + \cdots\right) \\ &= 1 + L_1(z-1) + \frac{L_2}{2!}(z-1)^2 + \cdots \end{aligned} \tag{4.1}$$

where the right member, because of (3.4) is absolutely convergent for $|z - 1| \leqq 1$. The coefficients on the right side of (4.1) are given by

$$L_n = \mu_{(n)} - n\mu_{(n-1)}\lambda + \frac{n(n-1)}{2!}\mu_{(n-2)}\lambda^2 - \cdots + (-1)^n\lambda^n \tag{4.2}$$

and the factorial moments may also be expressed by

$$\mu_{(n)} = L_n + nL_{n-1}\lambda + \frac{n(n-1)}{2!}L_{n-2}\lambda^2 + \cdots + \lambda^n. \tag{4.3}$$

These relations are readily derived by expressing (4.1) symbolically as

$$e^{-\lambda(z-1) + \mu(z-1)} = e^{L(z-1)} \tag{4.4}$$

where after expansion μ^n and L^n are to be replaced by $\mu_{(n)}$ and L_n respectively. (Cf. Jordan [7], p. 39). From (4.1) and (3.4) there is now derived

$$\varphi(z) = e^{\lambda(z-1)}\left(1 + L_1(z-1) + \frac{L_2}{2!}(z-1)^2 + \cdots\right). \tag{4.5}$$

Since the right member of (4.5) is absolutely and uniformly convergent for $|z - 1| \leqq 1$ for arbitrary λ, it may be expressed as

$$\varphi(z) = \left(1 + L_1 \frac{\partial}{\partial\lambda} + \frac{L_2}{2!}\frac{\partial^2}{\partial\lambda^2} + \cdots\right) e^{\lambda(z-1)}. \tag{4.6}$$

Since the radius of convergence of the right member of (4.6) is some value R_2 such that $|z - 1| < R_2 > 1$, it may be expressed as a power series about $z = 0$, or

$$\varphi(z) = \left(1 + L_1 \frac{\partial}{\partial\lambda} + \frac{L_2}{2!}\frac{\partial^2}{\partial\lambda^2} + \cdots\right) e^{-\lambda}\left(1 + \lambda z + \frac{\lambda^2 z^2}{2!} + \cdots\right). \tag{4.7}$$

Recalling now the definition of $\varphi(z)$ as given in (3.1), there is obtained by equating coefficients of like powers of z in (3.1) and (4.7)

$$p(r) = \left(1 + L_1 \frac{\partial}{\partial\lambda} + \frac{L_2}{2!}\frac{\partial^2}{\partial\lambda^2} + \cdots\right) \frac{e^{-\lambda}\lambda^r}{r!}. \tag{4.8}$$

Since it may be readily shown that

$$\frac{\partial^n}{\partial\lambda^n}\frac{e^{-\lambda}\lambda^r}{r!} = (-1)^n\Delta^n\frac{e^{-\lambda}\lambda^r}{r!} \tag{4.9}$$

where

$$\Delta\frac{e^{-\lambda}\lambda^r}{r!} = \frac{e^{-\lambda}\lambda^r}{r!} - \frac{e^{-\lambda}\lambda^{r-1}}{(r-1)!}$$

and

$$\Delta^n\frac{e^{-\lambda}\lambda^r}{r!} = \Delta^{n-1}\frac{e^{-\lambda}\lambda^r}{r!} - \Delta^{n-1}\frac{e^{-\lambda}\lambda^{r-1}}{(r-1)!}$$

we may also write (4.8) as

$$p(r) = \frac{e^{-\lambda}\lambda^r}{r!} - L_1\Delta\frac{e^{-\lambda}\lambda^r}{r!} + \frac{L_2}{2!}\Delta^2\frac{e^{-\lambda}\lambda^r}{r!} - \frac{L_3}{3!}\Delta^3\frac{e^{-\lambda}\lambda^r}{r!} + \cdots. \tag{4.10}$$

5. Necessity. Assume that the function $p(r)$ of (2.1), for arbitrary λ, is given by the absolutely convergent series

$$p(r) = \left(1 + L_1 \frac{\partial}{\partial\lambda} + \frac{L_2}{2!}\frac{\partial^2}{\partial\lambda^2} + \cdots\right) \frac{e^{-\lambda}\lambda^r}{r!}. \tag{5.1}$$

Since $e^{-\lambda}\lambda^r/r!$ is continuous with respect to λ, there follows, where z is a complex variable and $|z| \leqq 1$

$$\sum_{r=0}^{\infty} z^r p(r) = \sum_{r=0}^{\infty}\frac{z^r e^{-\lambda}\lambda^r}{r!} + L_1\frac{\partial}{\partial\lambda}\sum_{r=0}^{\infty}\frac{z^r e^{-\lambda}\lambda^r}{r!} + \frac{L_2}{2!}\frac{\partial^2}{\partial\lambda^2}\sum_{r=0}^{\infty}\frac{z^r e^{-\lambda}\lambda^r}{r!} + \cdots$$

$$= e^{\lambda(z-1)}\left(1 + L_1(z-1) + \frac{L_2}{2!}(z-1)^2 + \cdots\right) \tag{5.2}$$

$$= 1 + M_1(z-1) + \frac{M_2}{2!}(z-1)^2 + \frac{M_3}{3!}(z-1)^3 + \cdots$$

where

$$M_n = L_n + nL_{n-1}\lambda + \frac{n(n-1)}{2!}L_{n-2}\lambda^2 + \cdots + \lambda^n. \tag{5.3}$$

From (5.2) it follows that

$$M_n = \mu_{(n)} \tag{5.4}$$

where $\mu_{(n)}$ is as defined in (3.3). Since (5.1) becomes for $r = 0, \lambda = 0$

$$1 - \mu_{(1)} + \frac{1}{2!}\mu_{(2)} - \frac{1}{3!}\mu_{(3)} + \cdots \tag{5.5}$$

the assumed absolute convergence implies that

$$1 + |\mu_{(1)}| + \frac{1}{2!}|\mu_{(2)}| + \frac{1}{3!}|\mu_{(3)}| + \cdots + \frac{1}{n!}|\mu_{(n)}| + \cdots \tag{5.6}$$

converges.

6. Remarks. Obrechkoff [11] shows that his result includes those of Pollaczek-Geiringer [12], Szegö [12] (p. 110) and Jacob [5]. His theorem states that if the function $p(r)$, $(r = 0, 1, 2, \cdots)$, satisfies the following conditions

$$\sum_{r=1}^{\infty} 2^r r^A \, | p(r) | \tag{6.1}$$

is convergent for each finite number A, and

$$\frac{(4\lambda)^n}{(n+1)!}\sum_{r=1}^{n}\frac{|p(r)|}{r}(e^{-\lambda}\lambda^r/r!)^{-1} \tag{6.2}$$

tends toward zero as n increases indefinitely then $p(r)$ may be expressed in a convergent Charlier Type B series.

Uspensky [18] shows that if

$$\sum_{r=0}^{\infty} z^r p(r) \tag{6.3}$$

has a radius of convergence $R > 2$ then $p(r)$ may be expressed in a convergent Charlier Type B series.

Schmidt [16] shows that a necessary and sufficient condition for the convergence is that the function $\varphi(z)$ defined as in (3.1) (he does not explicitly impose the condition (2.1) on $p(r)$) be regular inside the two circles $|z| < 1$ and $|z - 1| < 1$ and with all its derivatives is continuous on the peripheries also. In the case that $p(r) \geqq 0$, the condition (2.5) is stronger, in fact in this case Schmidt [16] shows that a necessary and sufficient condition is that

$$\lim_{r=\infty} p(r)2^r r^k = 0$$

for all integral $k \geqq 0$. If $p(r) \geqq 0$, then Uspensky's condition is only just enough stronger than Schmidt's to keep it from being sufficient.

If (6.1) is satisfied, or if (6.3) is satisfied then (3.1) is absolutely convergent for $|z| \leqq 2$. Therefore, the point $z = 2$ is contained in the circle of convergence of (3.2) or (3.4) which implies that

$$1 + |\mu_{(1)}| + \frac{1}{2!}|\mu_{(2)}| + \cdots + \frac{1}{n!}|\mu_{(n)}| + \cdots$$

converges.

It is deemed worthy of special mention to point out, as both Schmidt and Uspensky have done, the striking fact that the necessary and sufficient condition for the validity of (2.4) is independent of λ. This arbitrariness of λ enables us to dispose of it so as to obtain better convergence. Indeed if we set $\lambda = \mu_{(1)}$ then as is evident from (4.2) $L_1 = 0$.

7. Special cases. It is of interest to note that (4.8) is the Taylor expansion if $p(r) = e^{-\mu}\mu^r/r!$, ($r = 0, 1, 2, \cdots$), for then (4.2) becomes

$$L_n = (\mu - \lambda)^n \tag{7.1}$$

since for the Poisson Exponential Distribution $e^{-\mu}\mu^r/r!$, ($r = 0, 1, 2, \cdots$), $\mu_{(n)} = \mu^n$ and (4.8) is then

$$\frac{e^{-\mu}\mu^r}{r!} = \frac{e^{-\lambda}\lambda^r}{r!} + (\mu - \lambda)\frac{\partial}{\partial\lambda}\frac{e^{-\lambda}\lambda^r}{r!} + \frac{(\mu - \lambda)^2}{2!}\frac{\partial^2}{\partial\lambda^2}\frac{e^{-\lambda}\lambda^r}{r!} + \cdots. \tag{7.2}$$

If $p(r)$ is finite, that is if $p(r) = 0$ for $r \geq n + 1$ then $\mu_{(k)} = 0$ for $k \geq n + 1$. Thus, for a finite function the condition (2.5) is satisfied.

8. Factorial moments. For functions $p(r)$, ($r = 0, 1, 2, \cdots$), satisfying (2.5), there may be derived from (3.1) and (3.4) the relation

$$r!p(r) = \mu_{(r)} - \mu_{(r+1)} + \frac{1}{2!}\mu_{(r+2)} - \frac{1}{3!}\mu_{(r+3)} + \cdots, \quad (r = 0, 1, 2, \cdots), \tag{8.1}$$

since each side is $\varphi^{(r)}(0)$ derived respectively from (3.1) and (3.4). It should be noted that for $\lambda = 0$ (4.5) leads to (8.1) rather than (4.8) so that (8.1) may be considered as the Charlier Type B series for $\lambda = 0$. The result (8.1) was derived for finite functions by Steffensen [17]. (Also compare Kaplansky [8]). This may also be expressed symbolically by

$$p(r) = \mu^r e^{-\mu}/r!, \qquad (r = 0, 1, 2, \cdots), \tag{8.2}$$

where after expansion μ^n is to be replaced by $\mu_{(n)}$. It is of interest to note the relation between the symbolic expression for $p(r)$ as a Poisson Exponential in (8.2) and the series (4.8), for (4.8) may be expressed symbolically as

$$\begin{aligned} p(r) &= e^{L(\partial/\partial\lambda)} \cdot \frac{e^{-\lambda}\lambda^r}{r!} = e^{-(\lambda+L)}(\lambda + L)^r/r! \\ &= \mu^r e^{-\mu}/r! \end{aligned} \tag{8.3}$$

since $e^{a(d/dx)}f(x) = f(x + a)$ and the relations (4.2), (4.3), (4.4).

9. Illustrations. Consider the function

$$p(r) = 1/2^{r+1}, \qquad (r = 0, 1, 2, \cdots). \tag{9.1}$$

For this function

$$\varphi(z) = \sum_{r=0}^{\infty} z^r p(r) = 1/(2 - z) \tag{9.2}$$

and

$$\varphi^{(n)}(1) = \mu_{(n)} = n! \tag{9.3}$$

so that (2.5) becomes

$$1 + 1 + 1 + \cdots \tag{9.4}$$

which does not converge. (It may be of interest to note that for this case (8.1) yields

$$p(0) = 1 - 1 + 1 - 1 + 1 - \cdots . \tag{9.5}$$

The series on the right in (9.5) is not convergent but is summable C_1 to $\frac{1}{2}$. For the latter see for example R. P. Agnew, [19].) In this case the first several coefficients of (4.8) are for $\lambda = 1$,

$$\begin{aligned} &L_1 = 0, \quad \frac{L_2}{2!} = .5000, \quad \frac{L_3}{3!} = .3333, \quad \frac{L_4}{4!} = .3750 \\ &\frac{L_5}{5!} = .3667, \quad \frac{L_6}{6!} = .3681, \quad \frac{L_7}{7!} = .3679, \quad \cdots . \end{aligned} \tag{9.6}$$

Let us now consider the function

$$p(0) = \tfrac{1}{2}, \quad p(r) = 1/3^r, \qquad (r = 1, 2, \cdots). \tag{9.7}$$

For this function

$$\varphi(z) = \sum_{r=0}^{\infty} z^r p(r) = \tfrac{1}{2} + \frac{z}{3-z} \tag{9.8}$$

and

$$\varphi^{(n)}(1) = \mu_{(n)} = \frac{n!}{2^n}\binom{3}{2}, \qquad (n = 1, 2, \cdots), \tag{9.9}$$

so that (2.5) becomes

$$1 + \binom{3}{2}\frac{1}{2} + \binom{3}{2}\frac{1}{2^2} + \binom{3}{2}\frac{1}{2^3} + \cdots \tag{9.10}$$

which converges. For this case (8.1) yields

$$(9.11)\qquad \begin{aligned} p(0) &= 1 - \left(\frac{3}{2}\right)\frac{1}{2} + \left(\frac{3}{2}\right)\frac{1}{2^2} - \left(\frac{3}{2}\right)\frac{1}{2^3} + \cdots = \tfrac{1}{2} \\ p(1) &= \left(\frac{3}{2}\right)\frac{1}{2} - 2!\left(\frac{3}{2}\right)\frac{1}{2^2} + \frac{3!}{2!}\left(\frac{3}{2}\right)\frac{1}{2^3} - \cdots = \tfrac{1}{8} \end{aligned}$$

etc.

In this case, the first several coefficients of (4.8) are for $\lambda = 0.75$

$$(9.12)\qquad \begin{aligned} &L_1 = 0, \quad \frac{L_2}{2!} = .093750, \quad \frac{L_3}{3!} = .046875, \quad \frac{L_4}{4!} = .019043 \\ &\frac{L_5}{5!} = .010840, \quad \frac{L_6}{6!} = .005173, \quad \frac{L_7}{7!} = .002622, \quad \cdots . \end{aligned}$$

Let us now consider the function (suggested by Prof. C. Wexler)

$$(9.13)\qquad p(0) = \frac{5}{3}, \qquad p(r) = (-1)^r \frac{5}{3}\left(\frac{2}{3}\right)^r, \qquad (r = 1, 2, \cdots).$$

For this function

$$(9.14)\qquad \sum_{r=0}^{\infty} p(r) = 1, \qquad \sum_{r=0}^{\infty} | p(r) | = 5$$

$$(9.15)\qquad \varphi(z) = \sum_{r=0}^{\infty} z^r p(r) = 5/(3 + 2z)$$

$$(9.16)\qquad \varphi^{(n)}(1) = \mu_{(n)} = (-1)^n n! (2/5)^n.$$

In this case (2.5) becomes

$$(9.17)\qquad 1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \cdots$$

which converges and (8.1) yields

$$(9.18)\qquad \begin{aligned} p(0) &= 1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \cdots = 5/3 \\ p(1) &= -2/5 - 2!(2/5)^2 - \frac{3!}{2!}(2/5)^3 - \cdots = -\frac{5}{3}\cdot\frac{2}{3} \end{aligned}$$

etc.

Note that for this case (6.1) or (6.3) are *not* satisfied. Using $\lambda = 1$, it is found that

$$(9.19)\qquad L_1 = -1.4, \quad \frac{L_2}{2!} = 1.06, \quad \frac{L_3}{3!} = -.5906, \quad \frac{L_4}{4!} = .2779, \quad \cdots .$$

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