with

$$W_i = \tfrac{1}{2}[(x_i - x_0)^2(\xi_i - x_0)f_{xxx}(\xi_i', \eta_i') + (x_i - x_0)^2(\eta_i - y_0)f_{xxy}(\xi_i', \eta_i') + 2(x_i - x_0)(y_i - y_0)f_{xy}(\xi_i, \eta_i) + (y_i - y_0)^2 f_{yy}(\xi_i, \eta_i)].$$

Corresponding formulas can be derived in this way for any value of n; in fact, several alternatives may be obtained in each case. In all cases the error $f(x_0, y_0)$ is given in terms of the derivatives of g alone if a polynomial of a certain type is used for the interpolating function. For equation (4), the suitable polynomial would be $h(x, y) = a + bx + cy$; for (5), $h(x, y) = a + bx + cy + dx^2 + exy + fy^2$; for (6), $h(x, y) = a + bx + cy + dx^2$. If the interpolating function $h(x, y)$ is not so chosen, the formulas remain valid, but derivatives of h will appear.

The same procedure is applicable to functions of any number of independent variables.

ON A LEMMA BY KOLMOGOROFF

By Kai-Lai Chung

Princeton University

The following lemma was proved by Kolmogoroff [1]:

If $e_1, e_2, \cdots, e_n$ are independent events and U an arbitrary event such that ($W(X)$ denoting the probability of X and $W_e(X)$ the conditional probability of X under the hypothesis of e)

$$W_{e_k}(U) \geqq u, \qquad W(e_1 + \cdots + e_n) \geqq u.$$

Then

$$W(U) \geqq \tfrac{1}{9}u^2.$$

This result seems of some interest in itself and may also have practical applications, for it is easily seen that [2] in general if $e_1, e_2, \cdots, e_n$ are arbitrary no information about $W_{e_1+\cdots+e_n}(U)$ can be obtained from that about $W_{e_k}(U)$, $k = 1, \cdots, n$. From this point of view the constant 1/9 is interesting, though it is unimportant in Kolmogoroff's proof of the law of large numbers. Using his original method this constant can easily be improved to 1/8. However, the following method will give a better result. At the same time we shall put it into a more general form.

Let

$$W_{e_k}(U) \geqq \alpha, \qquad \sum_{k=1}^{n} W(e_k) \geqq \beta.$$

Then we have for $1 \leqq k \leqq n$,

$$W(U) \geqq W(U(e_1 + \cdots e_k)) = W(Ue_1 + \cdots + Ue_k). \tag{1}$$

Now a simple case of certain inequalities due to Bonferroni and Frechet [3] states that for arbitray events $E_1, \cdots, E_k$ we have

$$W(E_1 + \cdots + E_k) \geqq \sum_{i=1}^{k} W(E_i) - \sum_{1 \leq i < j \leq k} W(E_i E_j). \tag{2}$$

Applying this to (1), we obtain

$$\begin{aligned} W(U) &\geqq \sum_{i=1}^{k} W(Ue_i) - \sum_{1 \leq i < j \leq k} W(Ue_i e_j) \\ &\geqq \sum_{i=1}^{k} W(e_i) W_{e_i}(U) - \sum_{1 \leq i < j \leq k} W(e_i) W(e_j), \end{aligned}$$

using the independence of $e_1, \cdots, e_k$. Hence

$$W(U) \geqq \alpha \sum_{i=1}^{k} W(e_i) - \frac{1}{2}\left(\sum_{i=1}^{k} W(e_i)\right)^2 + \frac{1}{2}\sum_{i=1}^{k} W^2(e_i).$$

By Cauchy's inequality,

$$\sum_{i=1}^{k} W^2(e_i) \geqq \frac{1}{k}\left(\sum_{i=1}^{k} W(e_i)\right)^2.$$

Writing $\sum_k = \sum_{i=1}^{k} W(e_i)$, we have

$$W(U) \geqq \left[\alpha - \left(\frac{1}{2} - \frac{1}{2k}\right)\sum_k\right]\sum_k . \tag{3}$$

Now let $0 < \gamma < \gamma_0 \leq 1$ where γ and γ_0 are to be determined later. If there is an e_i, $1 \leq i \leq n$ such that $W(e_i) \geqq \gamma\beta$, then

$$W(U) \geqq W(Ue_i) = W(e_i) W_{e_i}(U) \geqq \gamma\alpha\beta. \tag{4}$$

If every $W(e_i) < \gamma\beta$, we determine $k(> 1)$ such that

$$\Sigma_{k-1} < \gamma_0\beta \leqq \Sigma_k ;$$

thus

$$\Sigma_k < \Sigma_{k-1} + \gamma\beta < (\gamma_0 + \gamma)\beta.$$

And (3) yields

$$W(U) \geqq \left[\alpha - \frac{1}{2}\left(1 - \frac{1}{k}\right)(\gamma_0 + \gamma)\beta\right]\gamma_0\beta. \tag{5}$$

Now we choose γ so that the last terms in (4) and (5) be equal. This gives

$$\gamma = \frac{2\alpha - \left(1 - \frac{1}{k}\right)\gamma_0\beta}{2\alpha + \left(1 - \frac{1}{k}\right)\gamma_0\beta}\gamma_0.$$

To maximize γ, we put $\frac{d\gamma}{d\gamma_0} = 0$ and find

$$\gamma_0 = \frac{2(\sqrt{2} - 1)\alpha}{\beta}.$$

If $2(\sqrt{2} - 1)\alpha \leqq \beta$, this choice of γ_0 is admissible, and we obtain

$$\gamma = \frac{2 - \sqrt{2} + \frac{1}{k}(\sqrt{2} - 1)}{\sqrt{2} - \frac{1}{k}(\sqrt{2} - 1)} \frac{2(\sqrt{2} - 1)\alpha}{\beta}.$$

Thus we get (the first inequality being retained for small values of n)

$$(6) \qquad W(U) \geqq \frac{2 - \sqrt{2} + \frac{1}{n}(\sqrt{2} - 1)}{\sqrt{2} - \frac{1}{n}(\sqrt{2} - 1)} 2(\sqrt{2} - 1)\alpha^2$$

$$\geqq 2(\sqrt{2} - 1)^2\alpha^2 > \tfrac{34}{100}\alpha^2.$$

In case $2(\sqrt{2} - 1)\alpha > \beta$, we choose $\gamma_0 = 1$, and we obtain

$$\gamma = \frac{2\alpha - \left(1 - \frac{1}{k}\right)\beta}{2\alpha + \left(1 - \frac{1}{k}\right)\beta}.$$

Thus we get

$$W(U) \geqq \frac{2\alpha - \left(1 - \frac{1}{n}\right)\beta}{2\alpha + \left(1 - \frac{1}{n}\right)\beta}\alpha\beta$$

$$\geqq \frac{2\alpha - \beta}{2\alpha + \beta}\alpha\beta.$$

If we write $\beta = \eta\alpha$, we have

$$(7) \qquad W(U) \geqq \frac{2 - \eta}{2 + \eta}\eta\alpha^2.$$

We summarize (6) and (7) in the following table:

β/α	$\geqq 2(\sqrt{2} - 1)$	$= \eta < 2\sqrt{2} - 1$
$W(U)$	$\geqq 2(\sqrt{2} - 1)^2\alpha^2$	$\geqq \dfrac{2 - \eta}{2 + \eta}\eta\alpha^2$

Thus for Kolmogoroff's case ($\eta = 1$) we have $W(U) \geqq \frac{1}{3}\alpha^2$.

REFERENCES

[1] A. KOLMOGOROFF, "Bemerkungen zu meiner Arbeit 'Über die Summen zufälliger Grössen'," *Math. Annalen*, Vol. 102 (1929), pp. 434–488.

[2] K. L. CHUNG, "On mutually favorable events," *Annals of Math. Stat.*, Vol. 13 (1942), pp. 338–349.

[3] M. FRÉCHET, *Les probabilitiés associeés à un système d'évènements compatibles et dépendents*, Première partie, Hermann, Paris, 1939, p. 59.

APPROXIMATE WEIGHTS

BY JOHN W. TUKEY

Princeton University

1. Summary. The greatest fractional increase in variance when a weighted mean is calculated with approximate weights is, quite closely, the square of the largest fractional error in an individual weight. The average increase will be about one-half this amount.

The use of weights accurate to two significant figures, or even to the nearest number of the form: 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 55, 60, 65, 70, 75, 80, 85, 90, or 95, that is to say, of the form 10(1)20(2)50(5)100 $\times$ 10^r can thus reduce efficiency by at most $\frac{1}{4}$ percent, which is negligible in almost all applications.

2. Proof. Let the optimum weights be W_i, $i = 1, 2, \cdots, n$, with $W_i \geq 0$, where it is convenient to choose the normalization $\Sigma W_i = 1$. Let σ^2 be the variance of $\Sigma W_i x_i$, then the variance of each x_i must be σ^2/W_i, and since this is a weighted mean, the means of the x_i are the same.

Let the approximate weights be $W_i(1 + \lambda\theta_i)$, where $0 < \lambda < 1$ and $|\theta_i| \leq 1$, $i = 1, 2, \cdots, n$. Thus λ is the largest fractional error which may be made in the situation considered. We need the weak requirement $\lambda < 1$! The approximately weighted mean is

$$\frac{\sum W_i(1 + \lambda\theta_i)x_i}{\sum W_i(1 + \lambda\theta_i)} = \sum W_i \frac{1 + \lambda\theta_i}{1 + \lambda\bar{\theta}},$$