which implies (c′), and a theorem of Scheffé [2] states that (c′) implies (c).[2] Finally, it is not hard to show that the condition

(d) $$\lim_{n\to\infty} f_n(x) = f_0(x) \quad \text{almost everywhere}$$

implies (c′) but not conversely.

Summing up, we arrive at the following complete set of implication relations among the various modes of convergence which we have considered:

(20) $$(d) \rightarrow (c'') \rightleftarrows (c') \rightleftarrows (c) \rightarrow (b') \rightleftarrows (b) \rightarrow (a).$$

REFERENCES

[1] H. CRAMÉR, *Mathematical Methods of Statistics*, Princeton Univ. Press. 1946, pp. 58–60.
[2] H. SCHEFFÉ, "A useful convergence theorem for probability distributions," *Annals of Math. Stat.*, Vol. 18 (1947), pp. 434–438.
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ON RANDOM VARIABLES WITH COMPARABLE PEAKEDNESS

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The quality of a distribution usually referred to as its peakedness has often been measured by the fourth moment of the distribution. It is known, however, that there is no definite connection between the value of the fourth moment and what one may intuitively consider as the amount of peakedness of a distribution.[1] In the present paper a definition of relative peakedness is proposed and it is shown that this concept has properties which may make it practically applicable.

DEFINITION. *Let Y and Z be real random variables and Y_1 and Z_1 real constants. We shall say that Y is more peaked about Y_1 than Z about Z_1 if the inequality*

$$P(|Y - Y_1| \geqq T) \leqq P(|Z - Z_1| \geqq T)$$

is true for all $T \geqq 0$.

If, for example, Y and Z are normal random variables with expectations Y_1 and Z_1 and standard deviations σ_y and σ_z, and if $\sigma_y < \sigma_z$, then Y is more peaked about Y_1 than Z about Z_1. Similarly, if Y is a random variable such that $P(Y < a) = P(Y > b) = 0$ for $a < b$, and if Z is the discrete random variable with $P(Z = a) = P(Z = b) = \frac{1}{2}$, then Y is more peaked about $\frac{1}{2}(a + b)$ than Z about the same point.

[2] Scheffé actually proves that (d) implies (c), but the Lebesgue convergence theorem on which his proof is based holds for convergence in measure (see e.g. [3]).

[1] I. Kaplansky, "A common error concerning kurtosis," *Am. Stat. Assn. Jour.*, Vol. 40 (1945), p. 259.

LEMMA. *Let Y_1, Y_2, Z_1, Z_2 be continuous random variables*[2] *with the probability densities $\varphi_1(Y_1), \varphi_2(Y_2), f_1(Z_1), f_2(Z_2)$ such that*

1^0. *Y_1 and Y_2 are independent, Z_1 and Z_2 are independent,*

2^0. *$\varphi_i(Y_i) = \varphi_i(-Y_i)$ for all Y_i, $f_i(Z_i) = f_i(-Z_i)$ for all Z_i, $(i = 1, 2)$,*

3^0. *$\varphi_2(Y_2)$ and $f_1(Z_1)$ are not-increasing functions for positive values of the variables,* and

4^0. *Y_i is more peaked about 0 than Z_i, for $i = 1, 2$.*

Let $Y = Y_1 + Y_2$ and $Z = Z_1 + Z_2$. Under these assumptions Y is more peaked about 0 than Z.

PROOF: Let $\Phi_i(y) = P(Y_i \leqq y)$, $F_i(z) = P(Z_i \leqq z)$, for $i = 1, 2$, be the cumulative probability functions. For any random variables Y_1, Y_2, Z_1, Z_2 (not necessarily continuous) which fulfil assumption 1^0 we have, for any T, the relationships

$$\begin{aligned}
P(Y \leqq T) - P(Z \leqq T) &= \int_{-\infty}^{\infty} [\Phi_1(T - s)d\Phi_2(s) - F_1(T - s)dF_2(s)] \\
&= \int_{-\infty}^{\infty} [\Phi_1(T - s) - F_1(T - s)]d\Phi_2(s) \\
&\qquad + \int_{-\infty}^{\infty} F_1(T - s)[d\Phi_2(s) - dF_2(s)] \\
&= \int_{-\infty}^{\infty} [\Phi_1(T - s) - F_1(T - s)]d\Phi_2(s) \\
&\qquad - \int_{-\infty}^{\infty} [\Phi_2(s) - F_2(s)]dF_1(T - s) \\
&= \int_{-\infty}^{\infty} [\Phi_1(T - s) - F_1(T - s)]d\Phi_2(s) \\
&\qquad + \int_{-\infty}^{\infty} [\Phi_2(T - s) - F_2(T - s)]dF_1(s) \\
&= I_1(T) + I_2(T),
\end{aligned}$$

where

$$\begin{aligned}
I_1(T) &= \int_{-\infty}^{\infty} [\Phi_1(T - s) - F_1(T - s)]d\Phi_2(s) \\
&= \int_{-\infty}^{\infty} [\Phi_1(-s) - F_1(-s)]d\Phi_2(T + s) \\
&= \int_{-\infty}^{0} + \int_{0}^{\infty} \\
&= \int_{0}^{\infty} \{[F_1(s) - \Phi_1(s)]d\Phi_2(T - s) \\
&\qquad + [\Phi_1(-s) - F_1(-s)]d\Phi_2(T + s)\},
\end{aligned}$$

etc.

[2] As defined e.g. in H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, 1946, p. 169.

If the random variables have distributions symmetrical about zero (assumption 2^0) this is equal to

$$\int_0^{+\infty} \{[P(Z_1 \leqq s) - P(Y_1 \leqq s)]dP(Y_2 \leqq T - s) + [P(Y_1 \leqq -s) - P(Z_1 \leqq -s)]dP(Y_2 \leqq T + s)\}$$

$$= \int_0^{+\infty} \{[1 - P(Z_1 > s) - 1 + P(Y_1 > s)]dP(Y_2 \leqq T - s) + [P(Y_1 \geqq s) - P(Z_1 \geqq s)]dP(Y_2 \leqq T + s)\}$$

$$= \int_0^{+\infty} \{[P(Y_1 \geqq s) - P(Z_1 \geqq s)]d[P(Y_2 \leqq T + s) + P(Y_2 \leqq T - s)] - [P(Y_1 = s) - P(Z_1 = s)]dP(Y_2 \leqq T - s)\},$$

and we obtain

$$I_1(T) = \int_0^{+\infty} [P(Y_1 \geqq s) - P(Z_1 \geqq s)]d[P(Y_2 \leqq T + s) + P(Y_2 \leqq T - s)] - \int_0^{+\infty} [P(Y_1 = s) - P(Z_1 = s)]dP(Y_2 \leqq T - s). \tag{1.1}$$

By an analogous argument one derives the equality

$$I_2(T) = \int_0^{+\infty} [P(Y_2 \geqq s) - P(Z_2 \geqq s)]dP(Z_1 \leqq T + s) + P(Z_1 \leqq T - s)] - \int_0^{+\infty} [P(Y_2 = s) - P(Z_2 = s)]dP(Z_1 \leqq T - s). \tag{1.2}$$

Making use of the assumption that Y_1, Y_2, Z_1, Z_2, are continuous random variables, we conclude that the second integrals in (1.1) and (1.2) are zero, and we may write

$$I_1(T) = \int_0^{+\infty} [P(Y_1 \geqq s) - P(Z_1 \geqq s)][\varphi_2(T + s) - \varphi_2(T - s)]ds, \tag{2.1}$$

$$I_2(T) = \int_0^{+\infty} [P(Y_2 \geqq s) - P(Z_2 \geqq s)][f_1(T + s) - f_1(T - s)]ds. \tag{2.2}$$

For $T \geqq 0$ we have, making use of assumption 3^0,

$$\varphi_2(T + s) - \varphi_2(T - s) \leqq 0 \text{ if } 0 \leqq s \leqq T$$

$$\varphi_2(T + s) - \varphi_2(T - s) = \varphi_2(s + T) - \varphi_2(s - T) \leqq 0 \text{ if } 0 \leqq T \leqq s,$$

and similarly

$$f_1(T + s) - f_1(T - s) \leqq 0 \text{ for all } T \geqq 0 \text{ and } s \geqq 0.$$

Since according to assumption 4^0 we also have

$$P(Y_1 \geqq s) - P(Z_1 \geqq s) \leqq 0$$

$$P(Y_2 \geqq s) - P(Z_2 \geqq s) \leqq 0 \text{ for } s \geqq 0,$$

both integrands in (2.1) and (2.2) are non-negative for all values of s, and we conclude

$$P(Y \leqq T) - P(Z \leqq T) = I_1(T) + I_2(T) \geqq 0,$$

and hence

$$P(Y \geqq T) - P(Z \geqq T) \leqq 0 \text{ for } T \geqq 0. \tag{3.1}$$

From assumption 2^0 one easily sees that Y and Z have symmetrical probability distributions. This together with (3.1) leads to

$$P(Y \geqq T) - P(Z \geqq T) = P(Y \leqq -T) - P(Z \leqq -T) \leqq 0,$$

and thus to

$$P(|Y| \geqq T) - P(|Z| \geqq T) \leqq 0 \text{ for } T \geqq 0.$$

As can be seen from (1.1) and (1.2), the assumptions of the Lemma, in particular the assumption that all variables are continuous and the assumption 3^0, are rather special sufficient conditions for Y being more peaked about 0 than Z.

THEOREM 1. *Let Y and Z be continuous random variables with probability densities $\varphi(Y)$ and $f(Z)$ such that*

1^0. $\varphi(-Y) = \varphi(Y)$ for all Y, $f(-Z) = f(Z)$ for all Z,

2^0. $\varphi(Y)$ and $f(Z)$ are not-increasing functions for positive values of the variables,

3^0. Y is more peaked about 0 than Z.

Let $Y_1, Y_2, \cdots, Y_n$ and $Z_1, Z_2, \cdots, Z_n$ be random samples of Y and Z, respectively, and $\bar{Y}_n = \frac{1}{n}\sum_{j=1}^{n} Y_j$, $\bar{Z}_n = \frac{1}{n}\sum_{j=1}^{n} Z_j$.—Then $\bar{Y}_n$ is more peaked about 0 than $\bar{Z}_n$.

PROOF. From the preceding Lemma one concludes by simple induction that $Y' = Y_1 + Y_2 + \cdots + Y_n$ as well as $Z' = Z_1 + Z_2 + \cdots + Z_n$ are continuous random variables with distributions symmetrical about zero and probability densities not-increasing for positive values of the variables, such that Y' is more peaked about 0 than Z'. From this the theorem follows immediately.

The conjecture that assumption 2^0 of Theorem 1 might be superfluous is incorrect as may be seen from the following example:

Let Y be any continuous random variable with a distribution symmetrical about zero and such that $P(|Y| > a) = 0$ for some $a > 0$. Let Z be the discrete random variable with $P(Z = -a) = P(Z = a) = \frac{1}{2}$. We have for $0 \leqq T \leqq a$

$$P(|Y| \geqq T) \leqq 1 = P(|Z| \geqq T),$$

hence Y is more peaked about 0 than Z. If Y_1, Y_2 and Z_1, Z_2 are random samples of size 2, we have

$$P(\bar{Z}_2 = -a) = P(\bar{Z}_2 = a) = \tfrac{1}{4}, \qquad P(\bar{Z}_2 = 0) = \tfrac{1}{2},$$

and thus

$$P(|\bar{Z}_2| \geqq T) = \tfrac{1}{2} \text{ for } 0 < T \leqq a.$$

The random variable $\bar{Y}_2$ is continuous, with a distribution symmetrical about zero, such that $P(|\bar{Y}_2| \leqq a) = 1$. There exists, therefore, a T_1 such that $0 < T_1 \leqq a$ and that $P(|\bar{Y}_2| \geqq T_1) = \frac{3}{4}$. It follows that

$$P(|\bar{Y}_2| \geqq T_1) = \tfrac{3}{4} > \tfrac{1}{2} = P(|\bar{Z}_2| \geqq T_1),$$

hence $\bar{Y}_2$ is not more peaked about zero than $\bar{Z}_2$. The random variable Z is discrete, but it can be approximated by a continuous random variable with a U-shaped probability density, so that all the probabilities will be modified only very slightly and $\bar{Y}_2$ still will not be more peaked than $\bar{Z}_2$. Nothing will change in this example if one assumes that Y fulfils condition 2^0 of Theorem 1.

THEOREM 2. *Let Y be a continuous random variable such that*

1^0. $\varphi(-Y) = \varphi(Y)$ *for all* Y,

2^0. $\varphi(Y)$ *is a not-increasing function for* $Y > 0$,

3^0. $P(|Y| > a) = 0$ *for some* $a > 0$.

Let $Y_1, Y_2, \cdots, Y_n$ be a random sample of size n and $\bar{Y}_n = \frac{1}{n}\sum_{j=1}^{n} Y_j$. *Then, for any $y \geqq 0$, we have*

$$P(|\bar{Y}_n| \geqq y) \leqq \Psi_n\left(\frac{y}{a}\right), \tag{4.1}$$

where

$$\Psi_n(t) = \frac{2}{n} \sum_{(n/2)(t+1)<k\leqq n} (-1)^k \binom{n}{k} \left[\frac{n}{2}(t+1) - k\right]^n. \tag{4.2}$$

PROOF. Let Z be the random variable with uniform distribution in the interval $-1 \leqq Z \leqq 1$. If $Z_1, Z_2, \cdots, Z_n$ is a random sample, then $Z' = Z_1 + Z_2 + \cdots + Z_n$ has the cumulative probability function[3]

$$P(Z' \leqq z) \begin{cases} = 0, & z < -n, \\ = \dfrac{1}{n!} \displaystyle\sum_{i \leqq (z+n)/2} (-1)^i \binom{n}{i} \left(\frac{z+n}{2} - i\right)^n, & -n \leqq z \leqq n, \\ = 1, & z > n, \end{cases}$$

and $\bar{Z}_n = \dfrac{z'}{n}$ has the cumulative probability function

$$P(\bar{Z}_n \leqq \zeta) \begin{cases} = 0, & \zeta < -1, \\ = \dfrac{1}{n!} \displaystyle\sum_{i \leqq (n/2)(\zeta+1)} (-1)^i \binom{n}{i} \left[\frac{n}{2}(\zeta + 1) - i\right]^n, & -1 \leqq \zeta \leqq 1, \\ = 1, & \zeta > 1. \end{cases}$$

[3] This expression is due to Laplace. For derivation and discussion, see: J. V. Uspensky, *Introduction to Mathematical Probability*, McGraw-Hill, 1937, p. 279, and Cramér, op. cit., p. 245.

Thus,

$$P(|\bar{Z}_n| \geqq t) = 2[1 - P(\bar{Z}_n \leqq t)]$$
$$= 2\left\{1 - \frac{1}{n!} \sum_{i \leqq (n/2)(t+1)} (-1)^i \binom{n}{i} \left[\frac{n}{2}(t+1) - i\right]^n\right\},$$

and in view of the identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (u - k)^n = n!$$

this becomes

$$P(|\bar{Z}_n| \geqq t) = \frac{2}{n!} \sum_{(n/2)(t+1) < k \leqq n} (-1)^k \binom{n}{k} \left[\frac{n}{2}(t+1) - k\right]^n = \Psi_n(t)$$

for $0 \leqq t \leqq 1$. The random variable $\frac{Y}{a}$ is obviously more peaked about zero than Z. Since $\frac{Y}{a}$ and Z fulfil the assumptions of Theorem 1, it follows that $\frac{\bar{Y}_n}{a}$ is more peaked about zero than $\bar{Z}_n$, that is

$$P\left(\left|\frac{\bar{Y}_n}{a}\right| \geqq t\right) \leqq P(|\bar{Z}_n| \geqq t) = \Psi_n(t) \quad \text{for} \quad t \geqq 0.$$

Setting $at = y$, one obtains (4.1).

For $n \rightarrow \infty$ the function $\Psi_n(t)$ approaches asymptotically the probability $P(|X| \geqq t\sqrt{3n})$ for the normalized normal random variable X.[4] For $n = 8$ one obtains the following values which indicate a good approximation:

t	.3998	.5254	.6711
$P(\lvert X \rvert \geqq t\sqrt{24})$	.05	.01	.001
$\Psi_8(t)$	.049	.0092	.0005.

For smaller values of n, $\Psi_n(t)$ can be easily computed.

A METHOD FOR OBTAINING RANDOM NUMBERS

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The need for large quantities of random numbers to be used in sample design, subsampling, and other statistical problems is well known. Tippett's [1] numbers have been widely used for these purposes, despite criticism directed at their lack of randomness. The following procedure may be of interest to those

[4] Cramér, op. cit., p. 245.