LIMITING DISTRIBUTION OF A ROOT OF A DETERMINANTAL EQUATION

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1. Summary. The exact distribution of a root of a determinantal equation when the roots are arranged in a monotonic order was obtained by S. N. Roy [3] in 1943. A different method for deriving the distribution of any one of these roots has been described by the author in [2]. In the present paper the limiting forms of these distributions are obtained. This paper gives a method by which the limiting distributions can be obtained without undergoing an inordinate amount of mathematical labor.

2. Introduction. If $x = \| x_{ij} \|$ and $x^* = \| x^*_{ij} \|$ are two p-variate sample matrices with n_1 and n_2 degrees of freedom and $S = \| xx' \|/n_1$ and $S^* = \|x^*x^{*\prime}\|/n_2$ are the covariance matrices which under the null hypothesis are independent estimates of the same population covariance matrix, then the joint distribution of the roots of the determinantal equation $| A - \theta(A + B) | = 0$, where $A = n_1S$ and $B = n_2S^*$, was obtained by Hsu [1] in 1939 and is

$$(1)\qquad R'(l, \mu, \nu) = \frac{\pi^{l/2} \prod_{i=1}^{l} \Gamma\left(\frac{l + \mu + \nu + i - 2}{2}\right)}{\prod_{i=1}^{l} \Gamma\left(\frac{\mu + i - 1}{2}\right) \Gamma\left(\frac{\nu + i - 1}{2}\right) \Gamma\left(\frac{i}{2}\right)} \prod_{i=1}^{l} (\theta_i)^{\mu/2-1} \prod_{i=1}^{l} (1 - \theta_i)^{\nu/2-1} \prod_{i<j} (\theta_i - \theta_j),$$

$$(0 \le \theta_l \le \theta_{l-1} \le \cdots \le \theta_1 \le 1),$$

where $l = \min(p, n_1)$, $\mu = | p - n_1 | + 1$ and $\nu = n_2 - p + 1$. The distribution density may be expressed as

$$(2)\qquad R(l, m, n) = c(l, m, n) \prod_{i=1}^{l} [\theta_i^m (1 - \theta_i)^n] \prod_{i<j} (\theta_i - \theta_j),$$

where $m = \mu/2 - 1$ and $n = \nu/2 - 1$.

3. Method. Let $\theta_i = \zeta_i/n$ in (2). The joint distribution reduces to

$$(3)\qquad \frac{c(l, m, n)}{n^{l+lm+l(l-1)/2}} \prod_{i=1}^{l} [\zeta_i^m (1 - \zeta_i/n)^n] \prod_{i<j} (\zeta_i - \zeta_j)\, d\zeta_1 \cdots d\zeta_l,$$

$$(0 \le \zeta_l \le \zeta_{l-1} \cdots \le \zeta_1 \le n).$$

As n tends to infinity the limit of (3) is

$$K(l, m) \prod_{i=1}^{l} \zeta_i^m \prod_{i<j} (\zeta_i - \zeta_j) e^{-\Sigma \zeta_i} \, d\zeta_i \,. \tag{4}$$

$$(0 \le \zeta_l \le \zeta_{l-1} \cdots \le \zeta_1 < \infty).$$

The value of $K(l, m)$ is

$$\lim_{n\to\infty} \frac{c(l, m, n)}{n^{l+lm+l(l-1)/2}}$$

$$= \lim_{n\to\infty} \frac{\pi^{l/2} \prod_{i=1}^{l} \Gamma\left(\frac{l + 2m + 2n + i + 2}{2}\right)}{\prod_{i=1}^{l} \Gamma\left(\frac{2m + i + 1}{2}\right) \Gamma\left(\frac{2n + i + 1}{2}\right) \Gamma(i/2) \cdot n^{l+lm+l(l-1)/2}}$$

$$= \frac{\pi^{l/2}}{\prod_{i=1}^{l} \Gamma\left(\frac{2m + i + 1}{2}\right) \Gamma(i/2)} \cdot \lim_{n\to\infty} \frac{\prod_{i=1}^{l} \Gamma\left(\frac{l + 2m + 2n + i + 2}{2}\right)}{\prod_{i=1}^{l} \Gamma\left(\frac{2n + i + 1}{2}\right) \cdot n^{l+lm+l(l-1)/2}}$$

By using Stirling's approximation for gamma functions and after simplification we get

$$\lim_{n\to\infty} \frac{\prod_{i=1}^{l} \Gamma\left(\frac{l + 2m + 2n + i + 2}{2}\right)}{\prod_{i=1}^{l} \Gamma\left(\frac{2n + i + 1}{2}\right) \cdot n^{lm+l(l+1)/2}} = 1.$$

Hence

$$K(l, m) = \frac{\pi^{l/2}}{\prod_{i=1}^{l} \Gamma\left(\frac{2m + i + 1}{2}\right) \Gamma(i/2)},$$

and therefore

$$\begin{aligned} K(2, m) &= 2^{2m+1}/\Gamma(2m + 2), \\ K(3, m) &= 2^{2m+3}/[\Gamma(m + 1)\Gamma(2m + 3)], \\ K(4, m) &= 2^{4m+5}/[\Gamma(2m + 2)\Gamma(2m + 4)], \\ K(5, m) &= 2^{4m+9}/[3\Gamma(m + 1)\Gamma(2m + 3)\Gamma(2m + 5)]. \end{aligned} \tag{5}$$

Let

$$G_{l,m}(x) = K(l, m) \int_{0 \le \zeta_l \le \zeta_{l-1} \cdots < \zeta_1 \le x} \prod_{i=1}^{l} \zeta_i^m \prod_{i<j} (\zeta_i - \zeta_j) e^{-\Sigma \zeta_i} \prod d\zeta_i . \tag{6}$$

It can easily be observed that

$$G_{l,m}(x) = \Pr(\zeta_1 \le x) = \lim_{n\to\infty} \Pr(n\theta_1 \le x) = \lim_{n\to\infty} \Pr\left(\theta_1 \le \frac{x}{n}\right).$$

Thus the limiting form of the distribution of the largest root can be obtained by integrating the density given in (4) according to the method described by the author in [2]. It is, however, observed that the mathematical labor is reduced considerably by adopting the following method.

Referring to the results of the exact distribution of the largest root given in [2], let $F_{l,m,n}(x) = (0, l, l-1, \cdots, 1, x; m, n)$; thus $F_{2,m,n}(x) = (0, 2, 1, x; m, n)$ and $F_{3,m,n}(x) = (0, 3, 2, 1, x; m, n)$. Then $c(l, m, n)F_{l,m,n}(x)$ is the probability that none of the roots θ_i exceeds x, and is thus the cumulative distribution function of the greatest root. We shall show that $\lim_{n\to\infty} c(l, m, n)F_{l,m,n}(x/n) = G_{l,m}(x)$. The reader is, however, asked to refer to [2] for the detailed explanation of the notations and certain mathematical operations used in this paper.

4. Limiting distribution of the largest root. We will derive the distribution of the largest root for $l = 2$ and 3 by the two methods. A straightforward method will be named A. A second method, which proves to be very simple and easy will be called Method B.

(a) $l = 2$

(i) METHOD A. We have,

$$\Pr(n\theta_1 \le x) = G_{2,m}(x) = K(2, m) \int_{0<\zeta_2<\zeta_1<x} (\zeta_1\zeta_2)^m(\zeta_1 - \zeta_2)e^{-(\zeta_1+\zeta_2)}\, d\zeta_1\, d\zeta_2$$

By using the method described in [2], we have

$$G_{2,m}(x) = K(2, m)\left\{\int_{0<\zeta_2<\zeta_1<x} - \int_{0<\zeta_1<\zeta_2<x} \zeta_2^m e^{-\zeta_2}\cdot\zeta_1^{m+1}e^{-\zeta_1}\, d\zeta_1\, d\zeta_2\right\},$$

$$= K(2, m)\{T_0^{m,x}(y, 1, x; m+1) - T_0^{m,x}(0, 1, y; m+1)\},$$

where

$$T_a^{m,b}g(y) = \int_a^b g(y)\cdot y^m e^{-y}\, dy,$$

and

$$(7)\,(a, 1, b; m+1) = \int_a^b \zeta^{m+1}e^{-\zeta}\, d\zeta = (a^{m+1}e^{-a} - b^{m+1}e^{-b}) + (m+1)(a, 1, b; m).$$

Hence,

$$G_{2,m}(x) = K(2, m)T_0^{m,x}[y^{m+1}e^{-y} - x^{m+1}e^{-x} + (m+1)(y, 1, x; m) + y^{m+1}e^{-y}$$
$$- (m+1)(0, 1, y; m)],$$
$$= K(2, m)T_0^{m,x}[2y^{m+1}e^{-y} - x^{m+1}e^{-x}],$$

as $T_0^{m,x}[(y, 1, x; m) - (0, 1, y; m)] = 0$.

Therefore

$$\lim_{n\to\infty} \Pr(n\theta_1 \le x) = G_{2,m}(x) = K(2, m) \tag{8}$$
$$\cdot \left\{2\int_0^x y^{2m+1}e^{-2y}dy - x^{m+1}e^{-x}\int_0^x y^m e^{-y}\,dy\right\}.$$

When $x = \infty$, $G_{2,m}(x) = 1$; hence $K(2, m) = 2^{2m+1}/\Gamma(2m + 2)$, the value given in (5).

Now we shall derive the result by Method B.

(ii) METHOD B.

$$F_{2,m,n}(x) = (0, 2, 1, x; m, n) = \frac{1}{m + n + 2}$$
$$\cdot \left\{2\int_0^x y^{2m+1}(1 - y)^{2n+1}\,dy - x^{m+1}(1 - x)^{n+1}\int_0^x y^m(1 - y)^n\,dy\right\},$$

a result given in [2].

Replacing x by x/n, we get

$$(0, 2, 1, x/n; m, n) = \frac{1}{m + n + 2}$$
$$\cdot \left\{2\int_0^{x/n} y^{2m+1}(1 - y)^{2n+1}\,dy - (x/n)^{m+1}(1 - x/n)^{n+1}\int_0^{x/n} y^m(1 - y)^n\,dy\right\};$$

also, letting $y = u/n$, we have

$$(0, 2, 1, x/n; m, n) = \frac{1}{(m + n + 2)n^{2m+2}} \tag{9}$$
$$\cdot \left\{2\int_0^x u^{2m+1}(1 - u/n)^{2n+1}\,du - x^{m+1}(1 - x/n)^{n+1}\int_0^x u^m(1 - u/n)^n\,du\right\}.$$

Thus

$$\lim_{n\to\infty} \Pr(n\theta_1 \le x) = \Pr(\theta_1 \le x/n) = \lim_{n\to\infty} c(2, m, n)(0, 2, 1, x/n; m, n),$$
$$= \frac{2^{2m+1}}{\Gamma(2m + 2)}\left\{2\int_0^x u^{2m+1}e^{-2u}\,du - x^{m+1}e^{-x}\int_0^x u^m e^{-u}\,du\right\},$$

which is the same as (8), obtained by Method A.

(b) $l = 3$.

(i) METHOD A. We have

$$\Pr(n\theta_1 \le x) = G_{3,m}(x) = K(3, m)\int_{0<\zeta_3<\zeta_2<\zeta_1<x} \Pi\zeta_i^m\Pi(\zeta_i - \zeta_j)e^{-\Sigma\zeta_i}\Pi\,d\zeta_i$$
$$= K(3, m)\int_{0<\zeta_3<\zeta_2<\zeta_1<x} (\zeta_1\zeta_2\zeta_3)^m e^{-(\zeta_1+\zeta_2+\zeta_3)}\{1, 2, 3\}\,d\zeta_1\,d\zeta_2\,d\zeta_3,$$

where $\{1, 2, 3\} = \zeta_1\zeta_2\{1, 2\} + \zeta_3\zeta_1\{3, 1\} + \zeta_2\zeta_3\{2, 3\}$, as given in [2].

Or

$$G_{3,m(x)} = K(3, m)\left\{\int_{0<\zeta_3<\zeta_2<\zeta_1<x} + \int_{0<\zeta_1<\zeta_3<\zeta_2<x} + \int_{0<\zeta_2<\zeta_1<\zeta_3<x} \zeta_3^m e^{-\zeta_3}(\zeta_1\zeta_2)^{m+1}e^{-(\zeta_1+\zeta_2)}\{1, 2\}\, d\zeta_1\, d\zeta_2\, d\zeta_3\right\},$$

$$= K(3, m)\{T_0^{m,x}(y, 2, 1, x; m + 1) + T_0^{m,x}(0, 2, y, 1, x; m + 1) + T_0^{m,x}(0, 2, 1, y; m + 1)\},$$

where

$$(a, 2, 1, b; m) = \int_{a<\zeta_2<\zeta_1<b} (\zeta_1\zeta_2)^m(\zeta_1-\zeta_2)e^{-(\zeta_1+\zeta_2)}\, d\zeta_1\, d\zeta_2.$$

We have already obtained

$$(0, 2, 1, x; m) = G_{2,m}(x)/K(2, m) = \left\{2\int_0^x y^{2m+1}e^{-2y}\, dy - x^{m+1}e^{-x}\int_0^x y^m e^{-y}\, dy\right\}$$

as given in (8).

We also need the following results which are obtained by the method described for $l = 2$.

$$(10) \qquad (a, 2, 1, b; m) = \left\{2\int_a^b u^{2m+1}e^{-2u}\, du - (a^{m+1}e^{-a} + b^{m+1}e^{-b})\int_a^b u^m e^{-u}\, du\right\},$$

and

$$(11) \qquad (a, 2, b, 1, c; m) = \left\{b^{m+1}e^{-b}\int_a^c u^m e^{-u}\, du - a^{m+1}e^{-a}\int_b^c u^m e^{-u}\, du - c^{m+1}e^{-c}\int_a^b u^m e^{-u}\, du\right\}.$$

Using these results we have

$$G_{3,m}(x) = K(3, m)T_0^{m,x}\left\{2\int_y^x u^{2m+3}e^{-2u}\, du - (y^{m+2}e^{-y} + x^{m+2}e^{-x})\int_y^x u^{m+1}e^{-u}\, du - y^{m+2}e^{-y}\int_0^x u^{m+1}e^{-u}\, du + x^{m+2}e^{-x}\int_0^y u^{m+1}e^{-u}\, du + 2\int_0^y u^{2m+3}e^{-2u}\, du - y^{m+2}e^{-y}\int_0^y u^{m+1}e^{-u}\, du\right\}.$$

Simplifying we get

$$(12) \quad \lim_{n\to\infty} \Pr(n\theta_1 \le x) = G_{3,m}(x) = K(3, m)\left\{2\int_0^x u^{2m+3}e^{-2u}\, du\int_0^x u^m e^{-u}\, du - 2\int_0^x u^{m+1}e^{-u}\, du\int_0^x u^{2m+2}e^{-2u}\, du - x^{m+2}e^{-x}\left[2\int_0^x u^{2m+1}e^{-2u}\, du - x^{m+1}e^{-x}\int_0^x u^m e^{-u}\, du\right]\right\},$$

where $K(3, m) = 2^{2m+3}/[\Gamma(2m + 1)\Gamma(2m + 3)]$.

(ii) METHOD B.

$$F_{3,m,n}(x) = (0, 3, 2, 1, x; m, n)$$

$$= \frac{1}{m+n+3}[2(0, 1, x; 2m+3, 2n+1)(0, 1, x; m, n)$$

$$-2(0, 1, x; m+1, n)(0, 1, x; 2m+2, 2n+1)$$

$$- (0, x; m+2, n+1)(0, 2, 1, x; m, n)],$$

a result given in [2].

Replacing x by x/n and putting u/n for the variate y of integration, we have,

$$F_{3,m,n}(x) = (0, 3, 2, 1; x/n; m, n) = \frac{1}{m+n+3}$$

$$\left\{\frac{2}{n^{3m+5}} \int_0^x u^{2m+3}(1-u/n)^{2n+1}\,du \int_0^x u^m(1-u/n)^n\,du - \frac{2}{n^{3m+5}}\right.$$

$$\int_0^x u^{m+1}(1-u/n)^n\,du \int_0^x u^{2m+2}(1-u/n)^{2n+1}\,du - \frac{x^{m+2}(1-x/n)^{n+1}}{n^{3m+4}(m+n+2)}$$

$$\left.\left[2\int_0^x u^{2m+1}(1-u/n)^{2n+1}\,du - x^{m+1}(1-x/n)^{n+1}\int_0^x u^m(1-u/n)^n\,du\right]\right\}.$$

Hence

$$\lim_{n\to\infty} \Pr(n\theta_1 \le x) = \lim_{n\to\infty} \Pr\left(\theta_1 \le \frac{x}{n}\right) = \lim_{n\to\infty} c(3, m, n)F_{3,m,n}(x)$$

$$= K(3, m)\left\{2\int_0^x u^{2m+3}e^{-2u}\,du \int_0^x u^m e^{-u}\,du - 2\int_0^x u^{2m+2}e^{-2u}\,du \int_0^x u^{m+1}e^{+u}\,du\right.$$

$$\left.- x^{m+2}e^{-x}\left[2\int_0^x u^{2m+1}e^{-2u}\,du - x^{m+1}e^{-x}\int_0^x u^m e^{-u}\,du\right]\right\},$$

where

$$K(3, m) = 2^{2m+3}/[\Gamma(m+1)\Gamma(2m+3)].$$

This result is the same as (12) obtained by Method A.

We have thus shown that Method B is applicable for obtaining the limiting forms of the distribution of the largest root and that it is much simpler as compared to the straightforward method called Method A here.

The limiting distributions for the largest root for $l = 4$ and 5 are listed below.

(c) $l = 4$.

$$\lim_{n\to\infty} \Pr(n\theta_1 \le x) = \lim_{n\to\infty} \Pr\left(\theta_1 \le \frac{x}{n}\right) = G_{4,m}(x)$$

$$= K(4, m)\left\{2\int_0^x u^{2m+5}e^{-2u}\,du \frac{G_{2,m}(x)}{K(2, m)} - 2\int_0^x u^{2m+4}e^{-2u}\,du\right.$$

$$\left[2\int_0^x u^{2m+2}e^{-2u}\,du - x^{m+2}e^{-x}\int_0^x u^m e^{-u}\,du + (m+2)\frac{G_{2,m}(x)}{K(2, m)}\right]$$

$$\left.+ 2\int_0^x u^{2m+3}e^{-2u}\,du \frac{G_{2,m+1}(x)}{K(2, m+1)} - x^{m+3}e^{-x}\frac{G_{3,m}(x)}{K(3, m)}\right\},$$

where

$$K(4, m) = 2^{4m+5}/[\Gamma(2m + 2)\Gamma(2m + 4)].$$

(d) $l = 5$.

$$\lim_{n\to\infty} \Pr(n\theta_1 \le x) = \lim_{n\to\infty} \Pr\left(\theta_1 \le \frac{x}{n}\right) = G_{5,m}(x)$$

$$= K(5, m) \Big\{ 2\int_0^x u^{2m+7} e^{-2u}\, du\, \frac{G_{3,m}(x)}{k(3, m)} - 2\int_0^x u^{2m+6} e^{-2u}\, du$$

$$\cdot\Big[2\int_0^x u^{2m+4} e^{-2u}\, du \int_0^x u^m e^{-u}\, du - 2\int_0^x u^{2m+3} e^{-2u}\, du$$

$$\cdot\int_0^x u^{m+1} e^{-u}\, du - x^{m+3} e^{-x} \frac{G_{2,m}(x)}{K(2, m)} + (m + 3)\frac{G_{3,m}(x)}{K(3, m)}\Big]$$

$$+ 2\int_0^x u^{2m+5} e^{-2u}\, du \Big\{ 2\int_0^x u^{2m+5} e^{-2u}\, du \int_0^x u^m e^{-u}\, du$$

$$- 2\int_0^x u^{2m+3} e^{-2u}\, du \int_0^x u^{m+2} e^{-u}\, du - x^{m+3} e^{-x}$$

$$\cdot\Big[2\int_0^x u^{2m+2} e^{-2u}\, du - x^{m+2} e^{-x}\int_0^x u^m e^{-u}\, du + (m + 2)\frac{G_{2,m}(x)}{K(2, m)}\Big]\Big\}$$

$$- 2\frac{G_{3,m+1}(x)}{K(3, m + 1)}\int_0^x u^{2m+4} e^{-2u}\, du - x^{m+4} e^{-x}\frac{G_{4,m}(x)}{K(4, m)}\Big\},$$

where

$$K(5, m) = 2^{4m+9}/[3\Gamma(m + 1)\Gamma(2m + 3)\Gamma(2m + 5)].$$

5. Limiting distribution of the smallest root. It was shown in [2] that the exact distribution of the smallest root can be obtained by using the relation

$$\Pr(\theta_l \le x) = 1 - \Pr(\theta_1 \le 1 - x \mid \nu, \mu).$$

This relation, however, does not help in obtaining the limiting distribution of the smallest root from that of the largest root. The limiting distribution of the smallest root can be obtained by the method illustrated below.

(a) $l = 2$.

The exact distribution of the smallest root θ_2 can be expressed as

$$\Pr(\theta_2 \le x) = c(2, m, n)\{(0, 2, 1, x; m, n) + (0, 2, x, 1, z; m, n)\},$$

where $z = 1$. Replacing x by x/n, we get

$$\Pr(\theta_2 \le x/n) = c(2, m, n)\{(0, 2, 1, x/n; m, n) + (0, 2, x/n, 1, z; m, n)\},$$

where

$$(0, 2, 1, x/n; m, n) = \frac{1}{m + n + 2}\Big[2\int_0^{x/n} y^{2m+1}(1 - y)^{2n+1}\, dy$$

$$- (0, x/n; m + 1, n + 1)\int_0^{x/n} y^m(1 - y)^n\, dy\Big],$$

and

$$(0, 2, x/n, 1, z; m, n) = \frac{1}{m+n+2}\Big[(0, x/n; m+1, n+1) \cdot \int_0^z y^m(1-y)^n\,dy - (0, z; m+1, n+1)\int_0^{x/n} y^m(1-y)^n\,dy\Big],$$

as obtained from (6) of [2].

The limiting distribution of θ_2 is

$$\text{(13)}\quad \Pr(\theta_2 \le x/n) = \lim_{n\to\infty} c(2, m, n)\{(0, 2, 1, x/n; m, n) + (0, 2, x/n, 1, z; m, n)\}.$$

Putting u/n for y, the variate of integration and allowing n to tend to infinity, we have

$$\lim_{n\to\infty} c(2, m, n)(0, 2, 1, x/n; m, n) = K(2, m)\left\{2\int_0^x u^{2m+1}e^{-2u}\,du - x^{m+1}e^{-x}\int_0^x u^m e^{-u}\,du\right\},$$

and

$$\lim_{n\to\infty} c(2, m, n)(0, 2, x/n, 1, z; m, n) = K(2, m)x^{m+1}e^{-x}\int_0^\infty u^m e^{-u}\,du = K(2, m)x^{m+1}e^{-x}\Gamma(m+1).$$

Substituting these results in (13) we have

$$\lim_{n\to\infty}\Pr(n\theta_2 \le x) = \lim_{n\to\infty}\Pr(\theta_2 \le x/n) = K(2, m)\left\{2\int_0^x u^{2m+1}e^{-2u}\,du - x^{m+1}e^{-x}\int_0^x u^m e^{-u}\,du + x^{m+1}e^{-x}\Gamma(m+1)\right\},$$

where

$$K(2, m) = 2^{2m+1}/[\Gamma(2m+2)].$$

(b) $l = 3$.

The exact distribution of the smallest root can be expressed as

$$\Pr(\theta_3 \le x) = c(3, m, n)[(0, 3, 2, 1, x; m, n) + (0, 3, 2, x, 1, z; m, n) + (0, 3, x, 2, 1, z; m, n)],$$

where $z = 1$.

Replacing x by x/n and allowing n to tend to infinity we have

$$\text{(14)}\quad \Pr(n\theta_3 \le x) = \lim_{n\to\infty} c(3, m, n)[(0, 3, 2, 1, x/n; m, n) + (0, 3, 2, x/n, 1, z; m, n) + (0, 3, x/n, 2, 1, z; m, n)].$$

The values of these components on the right hand side of the above equation are given below.

$$\lim_{n\to\infty} c(3, m, n)(0, 3, 2, 1, x/n; m, n) = G_{3,m}(x), \quad \text{given by} \quad (12),$$

$$\lim_{n\to\infty} c(3, m, n)(0, 3, 2, x/n, 1, z; m, n)$$

$$(15) \quad = K(3, m)\left\{\int_x^\infty u^m e^{-u}\,du\left[2\int_0^x u^{2m+3} e^{-2u}\,du - x^{m+2} e^{-x}\int_0^x u^{m+1} e^{-u}\,du\right] - x^{m+2} e^{-x}\left[2\int_0^x u^{2m+1} e^{-2u}\,du - x^{m+1} e^{-x}\int_0^x u^m e^{-u}\,du\right] + x^{m+2} e^{-x}\int_x^\infty u^{m+1} e^{-u}\,du \int_0^x u^m e^{-u}\,du - 2\int_x^\infty u^{m+1} e^{-u}\,du\int_0^x u^{2m+2} e^{-2u}\,du\right\},$$

and

$$\lim_{n\to\infty} c(3, m, n)(0, 3, x/n, 2, 1, z; m, n) = K(3, m)\left\{\int_0^x u^m e^{-u}\,du\left[2\int_x^\infty u^{2m+3} e^{-2u}\,du - x^{m+2} e^{-x}\int_x^\infty u^{m+1} e^{-u}\,du\right] - x^{m+2} e^{-x}\left[2\int_x^\infty u^{2m+1} e^{-2u}\,du - x^{m+1} e^{-x}\int_x^\infty u^m e^{-u}\,du\right] + x^{m+2} e^{-x}\int_0^x u^{m+1} e^{-u}\,du\int_x^\infty u^m e^{-u}\,du - 2\int_0^x u^{m+1} e^{-u}\,du\int_x^\infty u^{2m+2} e^{-2u}\,du\right\}.$$

Substituting in (14) we have,

$$\lim_{n\to\infty} \Pr(n\theta_3 \le x) = \{2^{2m+3}/[\Gamma(m+1)\Gamma(2m+3)]\} \cdot\left\{2\int_0^\infty u^{2m+3} e^{-2u}\,du\int_0^x u^m e^{-u}\,du + 2\int_0^x u^{2m+3} e^{-2u}\,du\int_x^\infty u^m e^{-u}\,du - 2\int_0^\infty u^{m+1} e^{-u}\,du\int_0^x u^{2m+2} e^{-2u}\,du - 2\int_0^x u^{m+1} e^{-u}\,du\int_x^\infty u^{2m+2} e^{-2u}\,du - 2x^{m+2} e^{-x}\int_0^\infty u^{2m+1} e^{-2u}\,du - 2x^{m+2} e^{-x}\int_0^x u^{2m+1} e^{-2u}\,du + x^{2m+3} e^{-2x}\left(\int_0^x u^m e^{-u}\,du + \int_0^\infty u^m e^{-u}\,du\right)\right\}.$$

Or,

$$\lim_{n\to\infty} \Pr(n\theta_3 \le x) = 2^{2m+3}/[\Gamma(m+1)\Gamma(2m+3)] \cdot\left\{\frac{\Gamma(2m+4)}{2^{2m+4}}\int_0^x u^m e^{-u}\,du + 2\int_0^x u^{2m+3} e^{-2u}\,du\int_x^\infty u^m e^{-u}\,du - 2\Gamma(m+2)\int_0^x u^{2m+2} e^{-2u}\,du - 2\int_0^x u^{m+1} e^{-u}\,du\int_x^\infty u^{2m+2} e^{-2u}\,du - \frac{\Gamma(2m+2)}{2^{2m+1}} x^{m+2} e^{-x} - x^{m+2} e^{-x}\int_0^x u^{2m+1} e^{-2u}\,du + \Gamma(m+1) x^{2m+3} e^{-2x} + x^{2m+3} e^{-2x}\int_0^x u^m e^{-u}\,du\right\}.$$

Thus we have seen that this method can be used for obtaining the limiting distribution of the smallest root for any value of l.

6. Limiting distribution of any intermediate root. The above method can also be used for obtaining the limiting distribution of any intermediate root. We shall give the distribution of θ_2 for $l = 3$. We have

$$\text{(16)} \quad \Pr(\theta_2 \leq x) = c(3, m, n)\{(0, 3, 2, 1, x; m, n) + (0, 3, 2, x, 1, z; m, n)\},$$

where $z = 1$.

The $\lim_{n\to\infty} c(3, m, n)(0, 3, 2, 1, x/n; m, n)$ and $\lim_{n\to\infty} c(3, m, n)(0, 3, 2, x/n, 1, z; m, n)$ are given by (12) and (15) respectively. Substituting these results in (16) and simplifying we get

$$\lim_{n\to\infty} \Pr(n\theta_2 \leq x) = \frac{2^{2m+3}}{\Gamma(m+1)\Gamma(2m+3)}\left\{2\int_0^\infty u^m e^{-u}\,du \cdot \int_0^x u^{2m+3} e^{-2u}\,du - 2\int_0^\infty u^{m+1} e^{-u}\,du \int_0^x u^{2m+2} e^{-2u}\,du - 4x^{m+2}e^{-x}\int_0^x u^{2m+1}e^{-2u}\,du + 2x^{2m+3}e^{-2x}\int_0^x u^m e^{-u}\,du + x^{m+2}e^{-x}\left[\int_x^\infty u^{m+1}e^{-u}\,du\int_0^x u^m e^{-u}\,du - \int_x^\infty u^m e^{-u}\,du \int_0^x u^{m+1}e^{-u}\,du\right]\right\},$$

Or,

$$\lim_{n\to\infty} \Pr(n\theta_2 \leq x) = \frac{2^{2m+3}}{\Gamma(m+1)\Gamma(2m+3)}\left\{2\Gamma(m+1)\int_0^x u^{2m+3}e^{-2u}\,du - 2\Gamma(m+2)\int_0^x u^{2m+2}e^{-2u}\,du - 4x^{m+2}e^{-x}\int_0^x u^{2m+1}e^{-2u}\,du + 2x^{2m+3}e^{-2x} \cdot \int_0^x u^m e^{-u}\,du + x^{m+2}e^{-x}\left[\int_x^\infty u^{m+1}e^{-u}\,du\int_0^x u^m e^{-u}\,du - \int_x^\infty u^m e^{-u}\,du\int_0^x u^{m+1}e^{-u}\,du\right]\right\}.$$

Thus the limiting distribution of any intermediate root can be obtained by the above method.

7. Further problems. The limiting distribution of the largest root is found to be very helpful in obtaining the distribution of the sum of roots when $m = 0$. This condition implies that when the results are applied to canonical correlations the numbers of variates in the two sets differ by unity. The distributions for the sum of roots have been derived under the above condition for $l = 2, 3$ and 4 and the results are being presented in the next paper of this series.

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