A NOTE ON WEIGHING DESIGN

By K. S. Banerjee

Pusa, Bihar, India

1. Efficiency of weighing designs given by a three-fourth replicate. In the June issue of the *Annals*, Kempthorne [1] approached the construction of the orthogonal matrix X through fractional replicates, the original treatment of which was given by Finney [2]. Reference has been made to the use of a three fourth replicate for weighing designs. Details for such designs have not been furnished as their efficiency is lower than for the designs given by the completely orthogonal matrix X. In a three fourth replicate the treatment combinations have to be chosen in a particular manner for a comparatively easier analytical treatment both from the point of view of agrobiological experiments as well as weighing designs. The variance of each of the estimates in such a case will be $\sigma^2/2^{n-1}$. As a matter of fact, in a weighing design given by a fractional replicate of the type of $(2^\beta - 1)/2^\beta$, $(\beta = 1, 2, \cdots n)$, of 2^n experiments, the estimate of the variance of each object is independent of the fraction used and is equal to $\sigma^2/2^{n-1}$, the same as above.

2. Construction of a three fourth replicate. Kempthorne mentions that a factorial design of fraction $\frac{3}{4}$ could be taken to consist of a $\frac{1}{2}$ replicate on the identity $I = ABC$ and a quarter replicate based on the identity

$$I = A = BC = ABC.$$

If the half replicate based on the identity $I = ABC$ be taken to consist of all the treatments corresponding to the minus signs of the treatment contrast ABC [3], the additional quarter replicate can be chosen in two different ways. When however the treatments corresponding to the minus signs of both A and BC are kept, omitting the treatments corresponding to the plus signs of A and BC, the three fourth replicate so obtained will have certain advantages, which will not be available if the quarter replicate to be added is chosen to consist of the treatments corresponding to the plus signs of A and BC.

3. Behavior of the contrasts in a three fourth replicate and the efficiency of the weighing designs. In general, if there are n treatments giving rise to 2^n treatment combinations and if the defining contrasts be chosen as

$$I = ACD = BDE = ABCE,$$

it will be necessary to omit the treatment combinations corresponding to the plus signs of both ACD and BDE, which will be 2^{n-2} in number. In the three fourth replicate so obtained, 2^n treatment effects (inclusive of the mean) will divide themselves into sets of 4 treatment contrasts each. One of the sets will be I, ACD, BDE and $ABCE$ and any other set will be formed by multiplying any treatment contrast by the defining set namely, I, ACD, BDE and $ABCE$. Only three contrasts out of four in a set will be independent, so that only one of the contrasts, preferably the one of the highest order interaction may be kept as an alias (in agrobiological experiments) of the remaining three and may therefore be omitted. Each of the four contrasts within a set will be orthogonal to each of the other contrasts in the remaining sets, but within a set the four contrasts will be non-orthogonal to one another. Though non-orthogonal, the normal equations will be of the systematic type[1] and the matrix $X'X$, taking any three contrasts out of each set of four, will take the following form:

$$
(1) \qquad \begin{bmatrix}
x & a & a & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\
a & x & a & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\
a & a & x & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\
0 & 0 & 0 & x & a & a & 0 & 0 & 0 & \cdot & \cdot & \cdot \\
0 & 0 & 0 & a & x & a & 0 & 0 & 0 & \cdot & \cdot & \cdot \\
0 & 0 & 0 & a & a & x & 0 & 0 & 0 & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & x & a & a & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & a & x & a & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & a & a & x & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix},
$$

where the order of the matrix $N = \frac{3}{4}2^n$ is of the form $3t(t = 2^{n-2})$ and $x = 3.\,2^{n-2}$, $a = -\frac{1}{2}2^n + \frac{1}{4}2^{n-2} = -2^{n-2}$. The value of the above determinant $= (x - a)^{2t}$ $(x + 2a)^t$ and that of the determinant suppressing the first row and the first column $= (x - a)^{2t-1}(x + a)(x + 2a)^{t-1}$. $a^{ii} = (x + a)/(x - a)(x + 2a) = 1/2^{n-1}$, substituting for x and a. The variance of each estimate will therefore be $\sigma^2/2^{n-1}$.

4. General case. When a fraction of the type $\alpha/2^\beta = (2^\beta - 1)/2^\beta$ is used, the treatment combinations corresponding to the plus signs of the β independent contrasts is omitted. Out of each set of 2^β treatment contrasts, only $\alpha = 2^\beta - 1$ will de independent and the matrix will then take a form like that of (1), where

$$x \;= [(2^\beta - 1)2^n]/2^\beta = 2^{n-\beta}(2^\beta - 1) \quad \text{and}$$

$$a \;= -\tfrac{1}{2}\,2^n + [(2^{\beta-1} - 1)2^n]/2^\beta = -2^{n-\beta},$$

$$a^{ii} = [x + (\alpha - 2)a]/(x - a)\,[x + (\alpha - 1)a] = (2\cdot 2^{n-\beta})/2^n 2^{n-\beta} = 1/2^{n-1}.$$

The variance of each estimate $= \sigma^2/2^{n-1}$, the same as before. When a completely orthogonalised matrix of the order $(\alpha 2^n)/2^\beta = 2^{n-\beta}(2^\beta - 1)$ is available, the variance of an estimate will be $\sigma^2/2^{n-\beta}(2^\beta - 1)$. The ratio of the two variances $= 2^{n-1}/(2^n - 2^{n-\beta}) = 2^{\beta-1}/(2^\beta - 1)$, which shows how the efficiency of the weighing design decreases with the increasing value of the fraction. When $\beta = 1$, i.e. in a half replicate, the efficiency is 100 percent. The value of the fraction is never less than $\frac{1}{2}$.

[1] The analysis of the data available from agrobiological experiments will not be cumbersome to a prohibitive extent as in many other experiments where non-orthogonality creeps in. The results of investigation in this direction have already been communicated for publication elsewhere.

5. Independence of the estimates given by L_N in a biased spring balance. Kempthorne mentions that although the optimum designs for the spring balance case suggested by Mood furnish somewhat smaller variance than what is given by fractional replicates, these designs have the disadvantage that the estimates are correlated, whereas the estimates furnished by fractional replicates are orthogonal. The designs furnished by fractional replicates take account of the bias and if the weighing operation corresponding to the bias is omitted (in case where the spring balance is free from bias), the resultant scheme will fail to give independent estimates and the variance factors will be of the same magnitude as in the optimum design L_N of Mood with the same number of weighings. Again, these optimum designs may also be made to furnish independent estimates when the designs are adjusted in the manner as suggested by Mood to suit a biased spring balance.

It is true that the design matrix L_3 given by

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

does not give independent estimates as such; but when it is assumed that the spring balance has a bias and the design matrix is modified as follows:

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \tag{2}$$

the estimates except that for the bias will be orthogonal to one another and the variance of the estimated weights will necessarily be larger in value.

Before proving the general case, we notice that when -1 is substituted for 0 in (2) above, the resultant scheme will be an orthogonalised matrix. This is true not only in this particular instance but will hold good also in general. The constitution will be clear when the method of construction of L_N from H_{N+1} is recalled.

The distribution of ones in L_N gives a special type of symmetrical balanced incomplete block design, where $r = k = \frac{1}{2}(b + 1)$ and $\lambda = \frac{1}{4}(b + 1)$, while the distribution of zeros gives the complementary design for which $r_0 = r - 1$, $k_0 = k - 1$ and $\lambda_0 = \lambda - 1$. Therefore when a row of zeros and a column of ones (in that order) is added to L_N, the matrix $X'X$ of the resultant scheme takes the following form:

$$\begin{bmatrix} N+1 & r & r & r & \cdots & r \\ r & r & \lambda & \lambda & \cdots & \lambda \\ r & \lambda & r & \lambda & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ r & \lambda & \lambda & \lambda & \cdots & r \end{bmatrix}. \tag{3}$$

Making use of the identities well known in the theory of balanced incomplete block designs and remembering the relationships, $2\lambda = r = k = \frac{1}{2}(N + 1)$,

(I) The value of the determinant of

$$X'X = (r - \lambda)^{N-1}[(N + 1)\{r + \lambda(N - 1)\} - r^2N] = (r - \lambda)^{N-1}[r + \lambda(N - 1)],$$

(II) The value of the determinant suppressing the first row and the first column $= (r - \lambda)^{N-1}[r + \lambda(N - 1)]$,

(III) The value suppressing the second row and the second column

$$= (r - \lambda)^{N-2}[(N + 1)\{r + \lambda(N - 2)\} - r^2(N - 1)]$$
$$= (r - \lambda)^{N-2}[r + \lambda(N - 1)],$$

(IV) The value suppressing the first row and the third column

$$= (r - \lambda)^{N-2}[r\{r + \lambda(N - 2)\} - r\lambda(N - 1)]$$
$$= r(r - \lambda)^{N-1},$$

(V) The value suppressing the second row and third column

$$= (r - \lambda)^{N-2}[\lambda(N + 1) - r^2]$$
$$= 0.$$

Hence, the reciprocal matrix of $X'X$ will be given by

$$[X'X]^{-1} = \begin{bmatrix} 1 & -1/k & -1/k & \cdots & -1/k \\ -1/k & 2/k & 0 & \cdots & 0 \\ -1/k & 0 & 2/k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1/k & 0 & 0 & \cdots & 2/k \end{bmatrix}. \quad (4)$$

Let Y' denote the column matrix of the results of the weighings, $y_0, y_1, \cdots, y_N$ and B' the column matrix of the estimates of the weights $b_0, b_1, \cdots b_N$. Then the estimates will be given by the equation

$$B' = [X'X]^{-1}X'Y'.$$

It is easy to see that all the rows except the first in $[X'X]^{-1}X'$ are orthogonal to one another. To explain this, let us take the design given by (2). Here

$$X' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then $[X'X]^{-1}X'$ will be of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/k & +1/k & +1/k & -1/k \\ -1/k & +1/k & -1/k & +1/k \\ -1/k & -1/k & +1/k & +1/k \end{bmatrix}.$$

In all the rows excepting the first, for every 0 and +1 in X', there will respectively be a $-1/k$ and a $+1/k$ in $[X'X]^{-1}X'$. It has been mentioned before that an orthogonal matrix is obtained when -1 is substituted for every 0 in X or X'. Hence, N rows (all except the first) of $[X'X]^{-1}X'$ will be orthogonal and these N rows will estimate the N weights in orthogonal linear combinations of $y_0, y_1 \cdots y_N$.

It has been mentioned before that the distribution of zeros in L_N gives the complementary design, for which $r_0 = r - 1$, $K_0 = k - 1$ and $\lambda_0 = \lambda - 1$. If to such a design, a row of ones and a column of ones (in that order) be added to suit the estimation of the weights in a biased spring balance, exactly a similar situation will be obtained and the estimates will be orthogonal. It can readily be seen that the design furnished by Yates to weigh seven light objects and a bias is an illustration of this kind. The scheme given by Yates is the complementary design of L_7 with an additional row and a column of ones added to L_7.

The sixteen combinations of ten objects, $a, b, c, d, e, f, g, h, k, l$ include 1, which corresponds to weighing with empty pans or, in other words, which is devoted to estimating the bias. When 1 is omitted, $X'X$ will be of the form

$$\begin{bmatrix} r & \lambda & \lambda & \cdots & \lambda \\ \lambda & r & \lambda & \cdots & \lambda \\ \lambda & \lambda & r & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda & \lambda & \lambda & & r \end{bmatrix},$$

where $r = 8$ and $\lambda = 4$. The above matrix $X'X$ is obviously of the same form as given by L_{15}.

By following exactly the same procedure as given above, it can easily be seen that when the weighing operation 1 is included in the weighing design, the solution of the normal equations will lead to independent estimates. The absence of each letter will be a 0 and the presence $a + 1$ in the design matrix and if -1 is substituted for every zero, the resultant matrix will be orthogonal. In some cases, however, the number of letters in all the combinations will not be the same, i.e. k will not be constant. In such a situation, k in (4) will take the value of r or of 2λ.

REFERENCES

[1] O. Kempthorne, "The factorial approach to the weighing problem," *Annals of Math. Stat.*, Vol. 19 (1948), pp. 238–245.

[2] D. J. Finney, "The fractional replication of factorial arrangements," *Annals of Eugenics*, Vol. 12 (1945), pp. 291–301.

[3] F. Yates, *Tech. Commun. Bur. Soil Sci. Harpenden* no. 35 (1937), p. 11.

[4] Harold Hotelling, "Some improvements in weighing and other experimental techniques," *Annals of Math. Stat.*, Vol. 13 (1944), pp. 297–306.

[5] K. Kishen, "On the design of experiments for weighing," *Annals of Math. Stat.*, Vol. 14 (1945), pp. 294–301.

[6] A. M. Mood, "On Hotelling's weighing problem," *Annals of Math. Stat.*, Vol. 17 (1946), pp. 432–446.

[7] R. L. Plackett and J. P. Burman, "The design of optimum multifactorial experiment," *Biometrika*, Vol. 33 (1946), pp. 305–325.