ON THE RANGE-MIDRANGE TEST AND SOME TESTS WITH BOUNDED SIGNIFICANCE LEVELS[1]

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1. Summary. This paper is divided into two parts. The significance tests investigated in Part I concern the population mean and are based on the quantity

$$[(\text{sample midrange})\text{-}(\text{hypothetical mean})]/(\text{sample range}).$$

The case in which the observations are a sample from a normal population is considered in detail. The tests investigated are summarized in Table 1. These tests are found to be very efficient for small samples (see Table 4, power efficiency is defined in section 3). An investigation of several extremely non-normal populations using the values of $D\alpha$ obtained for normality indicates that the significance level of the range-midrange test is not very sensitive to the requirement of normality for small samples (see Table 6). Also the tests of Table 1 can be applied without computation through the use of an easily constructed graph (see section 4). These properties suggest that the range-midrange test is preferable to the Student t-test and the analogue of the Student t-test using the sample range (see [1] and [2]) whenever the sample size is sufficiently small.

Use of the range-midrange test for the case of normality was proposed by E. S. Pearson in [3], where properties of the test were experimentally investigated for the normal and certain non-normal populations.

In Part II several significance tests for the mean are developed which have a specified significance level for the case of a sample from a normal population but whose significance level is bounded near the specified value under very general conditions, one of which is that the observations are from continuous symmetrical populations. Some of these tests are range-midrange tests. Table 2 contains a summary of the tests and their properties (x_i = ith largest observation, $i = 1, \cdots, n$; conditions (D) are given in section 7).

PART I. THE RANGE-MIDRANGE TEST

2. Introduction. In 1929 E. S. Pearson proposed using the range-midrange test for the case of a sample from a normal population (see [3]) and experimentally investigated some of its properties for sample sizes of 5 and 10 and significance levels of 2% and 10% (symmetrical tests). Using the constants (corresponding to the D_α in this paper) determined for the case of normality,

[1] This paper was presented to a joint meeting of the Institute of Mathematical Statistics and the American Mathematical Society at New Haven, Conn. in September, 1947. The results presented in this paper were obtained in the course of research conducted under the sponsorship of the Office of Naval Research. This research was performed while the author was at Princeton University.

significance level and power function properties of these four tests were experimentally investigated for several non-normal populations. The results of this empirical investigation indicated that the range-midrange test is very efficient for normality and not very sensitive to the assumption of normality if the sample size is sufficiently small.

This paper presents an analytical investigation of properties of the range-midrange test for $n = 2, 3, \cdots, 10$ and a wide range of significance levels. The results of this investigation confirm the contention that the range-midrange test is very efficient for normality and small samples; also an analytical investigation of how the significance level changes for the case of certain extremely non-normal populations furnishes results which agree with the contention that the range-midrange test is not very sensitive to the requirement of normality for sufficiently small samples.

In most cases the results presented in this paper are not directly comparable with those obtained by Pearson. It was possible, however, to obtain values of $D\alpha$, ($\alpha = 5\%, 1\%$; $n = 5, 10$), from the results presented in [3]; these values were found to be in close agreement with the corresponding values of Table 5.

3. Efficiency of range-midrange. The purpose of this section is to use the relations derived in section 6 to determine the power efficiencies of tests A, B and C (see Table 1) for $\alpha = 1\%, 5\%$ and $n = 2, \cdots, 10$. To do this the method of defining power efficiency given in [4] and [5] will be used. As shown in [5], it is sufficient to consider only test A; for any fixed n and α, tests A, B and C all have the same power efficiency (note that the significance level of test C is 2α).

For a normal population (unknown variance) the most powerful test of the one-sided alternative $\mu < \mu_0$ is the appropriate Student t-test. The procedure used in determining the power efficiency of test A consists in first computing the power function of test A for the given values of n and α; then the sample size of the corresponding Student t-test at this significance level is varied until the power function of the t-test is approximately equal to that of test A. The size sample (not necessarily integral) thus obtained for the t-test divided by n is called the power efficiency of test A for the given values of n and α. Intuitively the power efficiency of a test measures the percentage of the total available information per observation which is being utilized by that test.

Table 3 contains values of the power function for test A. These values were computed from equation (3) of section 6 by approximate integration.

The corresponding values of the power function for the Student t-test were found by using the normal approximation given in [6]. This approximation was used for fractional degrees of freedom. The sample sizes considered as well as the resulting power function values are listed in Table 3. A comparison of the power function values for the two types of tests furnishes the approximate power efficiencies listed in Table 3.

For $n = 2$, test A is itself a Student t-test. The power efficiency is therefore 100% for that sample size. This combined with Table 3 furnishes power efficiencies at the 1% level for $n = 2, 6, 8, 10$ and at the 5% level for $n = 2, 6, 10$. The approximate power efficiencies given in Table 4 for other values of n were obtained from these values by graphical interpolation.

Table 4 shows that the power efficiency for $\alpha = 1\%$ is very good for $n \leq 8$, while for $\alpha = 5\%$ the efficiency is good for $n \leq 6$.

TABLE 1

Summary of range-midrange tests

Definitions	Tests: Accept	Tests: If	Significance Level
Test based on sample of size n, $(2 \leq n \leq 10)$, from an arbitrary normal population. .... x_1 = smallest sample value. x_n = greatest sample value.	(A) $\mu < \mu_0$	$D < -D_\alpha$	α
μ = the mean of the normal population. μ_0 = given hypothetical mean value to be tested. .... $D = \dfrac{\text{(sample midrange)-(hypothetical mean)}}{\text{(sample range)}}$ $= [(x_n + x_1)/2 - \mu_0]/(x_n - x_1)$.	(B) $\mu > \mu_0$	$D > D_\alpha$	α
.... D_α = constant depending on n and α. Values of α versus D_α for $2 \leq n \leq 10$ and $\alpha = 5\%, 2.5\%, 1\%, 0.5\%$ are given in Table 5.	(C) $\mu \neq \mu_0$	$\|D\| > D_\alpha$	2α

4. Construction of graph. In most problems to which a test of the type developed in this paper would be applied, the values of the sample can be considered to have practical lower and upper limits, say a and b. For example, in many situations zero is a lower limit for the sample values. From a practical viewpoint these limits on the sample values do not contradict the assumption that the population is normal, since the area under that part of the normal distribution which lies outside the interval (a, b) can be considered negligible. Thus, since $\Pr(u/v^2 \lessgtr w) = \Pr(u \lessgtr v^2 w)$, test A can be restated in the form

Accept $\mu < \mu_0$ if the sample point (x_1, x_n) falls in the region (A) of the x_1, x_n

TABLE 2

Some one-sided and symmetrical tests with bounded significance levels

n	Tests: SYMMETRICAL: Accept $\mu \neq \mu_0$ if either		Significance Level for Normality		Significance Level Bounds for Conditions (D)				Approx. Efficiency for normality
	ONE-SIDED: Accept $\mu < \mu_0$ if	ONE-SIDED: Accept $\mu > \mu_0$ if	One-sided	Symmetrical	1-sided Tests Upper	1-sided Tests Lower	Symmet. Tests Upper	Symmet. Tests Lower	
			%	%	%	%	%	%	%
4	$1.055x_4 - .055x_1 < \mu_0$	$1.055x_1 - .055x_4 > \mu_0$	5	10	6.2		12.5		96
5	$.63x_5 + .37x_4 < \mu_0$	$.63x_1 + .37x_2 > \mu_0$	5	10	6.2	3.1	12.5	6.2	99
	$1.02x_5 - .02x_1 < \mu_0$	$1.02x_1 - .02x_5 > \mu_0$	2.5	5	3.1		6.2		97
6	$.63x_6 + .37x_5 < \mu_0$	$.63x_1 + .37x_2 > \mu_0$	2.5	5	3.1	1.6	6.2	3.1	98.5
	$1.06x_6 - .06x_1 < \mu_0$	$1.06x_1 - .06x_6 > \mu_0$	1	2	1.6		3.1		98
7	$.785x_7 + .215x_6 < \mu_0$	$.785x_1 + .215x_2 > \mu_0$	1	2	1.6	0.8	3.1	1.6	97
	$1.05x_7 - .05x_1 < \mu_0$	$1.05x_1 - .05x_7 > \mu_0$	0.5	1	0.8		1.6		96
8	$\max[x_7, (.5x_8 + .28x_6 + .22x_7)] < \mu_0$	$\min[x_2, (.5x_1 + .28x_3 + .22x_2)] > \mu_0$	Approx. 1	Approx. 2	1.2	0.8	2.3	1.6	98
	$.785x_8 + .215x_7 < \mu_0$	$.785x_1 + .215x_2 > \mu_0$	0.5	1	0.8	0.4	1.6	0.8	97
9	$\max[x_8, (.5x_9 + .28x_7 + .22x_8)] < \mu_0$	$\min[x_2, (.5x_1 + .28x_3 + .22x_2)] > \mu_0$	Approx. 0.5	Approx. 1	0.6	0.4	1.2	0.8	98.5

TABLE 3

Power function values for test A

Type Test	Sample Size	Approx. Efficiency	Significance Level	Approximate Values of Power Function				
				$\delta = \frac{1}{2}$	$\delta = 1$	$\delta = 1\frac{1}{2}$	$\delta = 2$	$\delta = 2\frac{1}{2}$
		%						
t	5.4		.05	.244	.607	.886	.969	
A	6	90	.05	.259	.599	.868	.967	
t	7.5		.05	.333	.783	.971		
A	10	75	.05	.351	.779	.962		
t	5.88		.01	.071	.248	.551	.820	.957
A	6	98	.01	.077	.271	.568	.809	.935
t	7.2		.01	.091	.371	.749	.949	
A	8	90	.01	.104	.389	.728	.923	
t	8		.01	.108	.453	.832	.976	
A	10	80	.01	.124	.462	.814	.963	

TABLE 4

Power efficiencies of tests A, B and C for $\alpha = 5\%$, 1% and $2 \leq n \leq 10$

α	n								
	2	3	4	5	6	7	8	9	10
.01	100%	99.7%	99.4%	99%	98%	95%	90%	85%	80%
.05	100%	98.5%	96%	93.5%	90%	86.5%	82.5%	78.5%	75%

TABLE 5

Approximate values of D_α for $\alpha = 5\%$, 2.5%, 1%, 0.5% and $2 \leq n \leq 10$

α	n								
	2	3	4	5	6	7	8	9	10
0.5%	31.83	3.02*	1.37*	.85*	.66	.55*	$.47_5$	$.42_5$	.39*
1%	15.91	2.11*	1.04*	.71	.56*	$.47_5$	.42*	.38	.35*
2.5%	6.35	1.30	.74	.52	.43	$.37_5$	.33	.30	$.27_5$
5%	3.16	.90*	$.55_5$*	$.42_5$	.35*	.30	$.26_5$	.24	$.22_5$*

* These values of D_α were verified directly by substitution and integration. The remaining values of D_α for $3 \leqq n \leqq 10$ were obtained from these and other values of D_α, ($\alpha \pm .005, .01, .025, .05$), by graphical interpolation.

plane defined by

$$(1/2 + D_\alpha)x_n + (1/2 - D_\alpha)x_1 < \mu_0, \qquad x_n \geq x_1, \qquad a \leq x_1, x_n \leq b.$$

TABLE 6

Effect of non-normality on the significance level of the range-midrange test

n	Probability Density Function	Significance Level											
		Test A				Test B				Test C			
	Normal	.05	.025	.01	.005	.05	.025	.01	.005	.10	.05	.02	.01
3	1 if $0 \leq x \leq 1$	.064	.039	.018	.010	.064	.039	.018	.010	.128	.078	.036	.020
4	0 otherwise	.053	.033	.017	.0096	.053	.033	.017	.0096	.106	.066	.034	.0192
5	Mean = $\frac{1}{2}$	.043	.029	.015	.0094	.043	.029	.015	.0094	.086	.058	.030	.0188
3	$\frac{1}{2}e^{\lvert x \rvert}$, $-\infty < x < \infty$	.036	.017	.0063	.0031	.036	.017	.0063	.0031	.072	.034	.0126	.0062
4	Mean = 0	.043	.016	.0055	.0024	.043	.016	.0055	.0024	.086	.032	.0101	.0048
5		.095	.026	.0059	.0027	.095	.026	.0059	.0027	.190	.052	.0118	.0054
3	$\frac{3}{2}x^2$ if $-1 \geq x \geq 1$	.119	.104	.073	.050	.119	.104	.073	.050	.238	.208	.146	.100
4	0 otherwise	.062	.061	.055	.045	.062	.061	.055	.045	.124	.122	.110	.090
5	Mean = 0	.031	.031	.031	.029	.031	.031	.031	.029	.062	.062	.062	.058
3	e^{-x} if $0 \leq x < \infty$	.014	.0067	.0025	.0012	.158	.108	.059	.035	.172	.115	.062	.036
4	0 otherwise	.013	.0048	.0016	.0007	.144	.104	.065	.042	.157	.109	.067	.043
5	Mean $\doteq$ 1	.017	.0055	.0013	.0006	.122	.096	.061	.045	.139	.102	.062	.046
3	$2x$ if $0 \leq x \leq 1$	.035	.019	.0075	.0038	.096	.061	.030	.017	.131	.080	.038	.021
4	0 otherwise	.031	.016	.0065	.0031	.083	.055	.031	.018	.114	.071	.038	.021
5	Mean = $\frac{2}{3}$	.028	.015	.0057	.0031	.068	.050	.028	.019	.096	.065	.032	.020
3	$3x^2$ if $0 \leq x \leq 1$	.027	.014	.0053	.0026	.112	.072	.037	.021	.139	.086	.042	.024
4	0 otherwise	.024	.011	.0043	.0019	.099	.067	.039	.024	.123	.078	.043	.026
5	Mean = $\frac{3}{4}$	.023	.012	.0037	.0019	.082	.061	.036	.025	.105	.073	.040	.027

Likewise test B can be restated as

Accept $\mu > \mu_0$ *if* (x_1, x_n) *falls in the region* (B) *defined by*

$$(1/2 - D_\alpha)x_n + (1/2 + D_\alpha)x_1 > \mu_0, \qquad x_n \geq x_1, \qquad a \leq x_1, x_n \leq b.$$

Test C now becomes

Accept $\mu \neq \mu_0$ *if* (x_1, x_n) *falls in either of the regions* (A) *or* (B).

Figure 1 (i) contains a schematic diagram of the regions (A) and (B). Test A can be applied by constructing a graph of the region (A) and giving the instructions to accept $\mu < \mu_0$ if (x_1, x_n) falls in (A). Similarly for test B and region (B). Test C is applied by constructing a graph of both (A) and (B) and accepting $\mu \neq \mu_0$ if (x_1, x_n) falls in either (A) or (B).

Frequently it is desirable to simultaneously consider more than one significance level. This can be accomplished in the manner indicated by Figure 1(ii).

5. Effect of non-normality on significance level. It has been shown that the range-midrange test compares very favorably with the Student t-test for sufficiently small samples and normality. In practice, however, it may happen that normality is assumed for cases in which the population is not even approximately normal. Although this represents an error in judgment on the part of the person applying the test, such situations will undoubtedly occur if the range-midrange test is used very frequently. The purpose of this section is to investigate the effect of non-normality on the significance level of the range-midrange test when the values of D_α based on normality are used. The corresponding effect of these non-normal populations on the significance level of the t-test was not considered because of computational difficulties; however the effect of some other non-normal populations on the significance level of the t-test was experimentally investigated by Pearson in [3]. The results of this empirical investigation and of later investigations shows that the significance level of the t-test is not very sensitive to the requirement of normality for small samples.

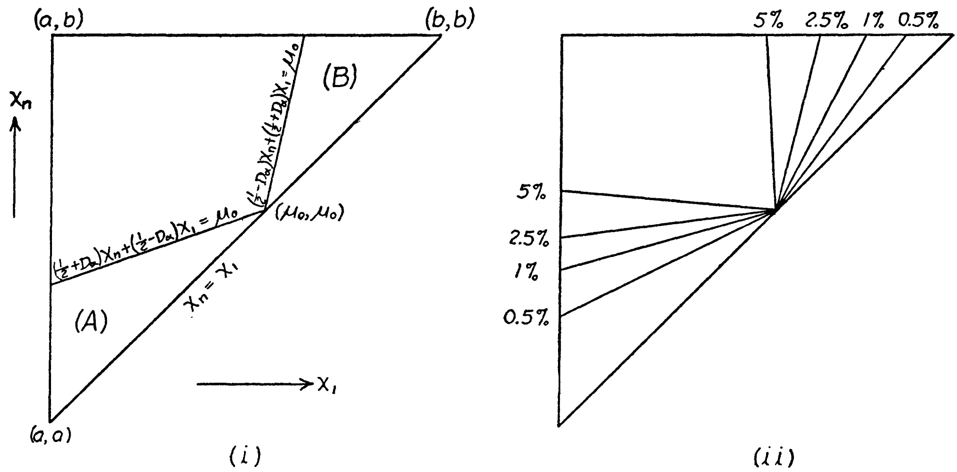

FIG. 1. SCHEMATIC DIAGRAMS OF REGIONS USED IN CONSTRUCTION OF GRAPHS

Six populations were chosen for investigation. Three of these populations are symmetrical while the remaining three are strongly asymmetrical. These particular populations were considered because their probability density functions have a wide variety of different shapes; also because the significance level of the range-midrange test can be computed in closed form for these populations.

The populations investigated are defined by their probability density functions. Table 6 contains a list of the probability density functions considered along with the resulting significance levels for the range-midrange test. The cases investigated are $n = 3, 4, 5$ and $\alpha = 5\%, 2.5\%, 1\%, 0.5\%$. Larger values of n were not used because of computational difficulties. The situation of $n = 2$ was not considered because the t-test and the range-midrange test are identical for this case. The significance levels of Table 6 were computed by making direct application of (1) and (2) of section 6.

6. Significance level and power function derivations. The purpose of this section is to present derivations of the significance level and power function expressions which were used in the preceding sections. First a general probability expression will be evaluated. Direct applications of the results obtained for this expression yield the required significance level and power function relations.

Let x_1 and x_n be the smallest and largest values, respectively, of a sample of size n drawn from a population with probability density function $f(x)$. The non-zero probability range of this population is $\gamma \leq x \leq \beta$. Also let three constants c_1, c_n, c_0, $(c_1 + c_n = 1)$, be given and consider the value of

$$\Pr (c_1 x_1 + c_n x_n < c_0); \quad \text{where} \quad M(z) = \int_{-\infty}^{z} f(y)\, dy.$$

Using direct methods it is found that the value of this expression is given by

$$[M(c_0)]^n \qquad \text{if } c_1 = 0.$$

$$0 \qquad \text{if } 0 < c_1 < 1,\ c_0 \leq \gamma$$

$$M\left(\frac{c_0 - c_1\gamma}{c_n}\right)^n - n\int_{c_0}^{(c_0 - c_1\gamma)/c_n} \left[M(V) - M\left(\frac{c_0 - c_n V}{c_1}\right)\right]^{n-1} f(V)\, dV \qquad \text{if } 0 < c_1 < 1, \quad c_0 > \gamma.$$

$$(1) \quad 1 - [1 - M(c_0)]^n \qquad \text{if } c_1 = 1$$

$$0 \qquad \text{if } c_1 > 1,\ c_0 \leq \min\,[\gamma, c_1\gamma + c_n\beta].$$

$$1 - n\int_{(c_0 - c_n\gamma)/c_n}^{\beta} \left[M(V) - M\left(\frac{c_0 - c_n V}{c_1}\right)\right]^{n-1} f(V)\, dV - M\left(\frac{c_0 - c_n\gamma}{c_n}\right)^n \qquad \text{if } c_1 > 1, \quad c_1\gamma + c_n\beta < c_0 \leq \gamma.$$

$$1 - n\int_{c_0}^{\beta} \left[M(V) - M\left(\frac{c_0 - c_n V}{c_1}\right)\right]^{n-1} f(V)\, dV \qquad \text{if } c_1 > 1, \quad c_0 > \gamma.$$

The value of $\Pr(c_1x_1 + c_nx_n < c_0)$ for $c_1 < 0$ can be obtained from the above results for $c_1 > 1$. It is easily shown that

$$\Pr(c_1x_1 + c_nx_n < c_0) = 1 - \Pr(c_1'y_1 - c_n'y_n < c_0'), \tag{2}$$

where

$$c_1' = c_n, \qquad c_n' = c_1, \qquad c_0' = -c_0,$$

and y_1, y_n are the smallest and largest values, respectively, of a sample of size n drawn from a population with probability function $g(y) = f(-y)$. Thus if $c_1 < 0$, $c_1' = c_n > 1$ and obvious modifications of the results for $c_1 > 1$ will furnish the value of $\Pr(c_1'y_1 + c_n'y_n < c_0')$.

The above general results were used in section 5 to investigate the effect of non-normality on the significance level of the range-midrange test.

Now consider the case in which the n sample values are drawn from a normal population with mean μ and variance σ^2. Then, for test A,

$$\begin{aligned}\text{Power Function} &= \Pr\{(1/2 - D_\alpha)x_1 + (1/2 + D_\alpha)x_n < \mu_0\} \\ &= \Pr\{(1/2 - D_\alpha)z_1 + (1/2 + D_\alpha)z_n < \delta\},\end{aligned}$$

where

$$z_1 = (x_1 - \mu)/\sigma, \quad z_n = (x_n - \mu)/\sigma, \quad \delta = (\mu_0 - \mu)/\sigma.$$

Using the above results with

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad M(z) = N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx,$$

it is found that the power function for test A is

$$\begin{cases} 1 - n \displaystyle\int_\delta^\infty \left[N(V) - N\left\{\frac{\delta - (1/2 + D_\alpha)V}{1/2 - D_\alpha}\right\}\right]^{n-1} f(V)\, dV & \text{if} \quad D_\alpha < 1/2; \\ [N(\delta)]^n \quad \text{if} \quad D_\alpha = 1/2; \\ n \displaystyle\int_{-\infty}^{\delta} \left[N(V) - N\left\{\frac{\delta - (1/2 + D_\alpha)V}{1/2 - D_\alpha}\right\}\right]^{n-1} f(V)\, dV, & \text{if} \quad D_\alpha > 1/2. \end{cases} \tag{3}$$

The value of D_α (for given n) corresponding to a specified significance level α for test A is obtained by solving the equation

$$\alpha = P_A(0), \tag{4}$$

where $P_A(\delta)$ is the power function for test A. From symmetry and the fact that test C is a combination of tests A and B, test B has significance level α and test C significance level 2α for this value of D_α.

For $n = 2$, test A becomes a Student t-test with one degree of freedom if D_α is replaced by $t_\alpha/2$. The relation $D_\alpha = t_\alpha/2$ gives an easily applied method of computing D_α for this case.

Approximate values of D_α for $\alpha = 5\%, 2.5\%, 1\%, 0.5\%$ are contained in

Table 5 for $2 \leq n \leq 10$. For $3 \leq n \leq 10$, these values were obtained from (3) and (4) by approximate integration and interpolation. For $n = 2$, the relation between D_α and t_α was used.

PART II. SOME TESTS WITH BOUNDED SIGNIFICANCE LEVELS

7. Introduction. In this part some significance tests (for the mean) are derived which are based on the assumption of a sample from a normal population. These tests have the property that the significance level is bounded near the value for normality under very general conditions. These conditions are

(D) (a) The observations used for a test are independent.
(b) Each observation comes from a continuous symmetrical population with mean μ.

It is to be emphasized that no two observations are necessarily drawn from the same population.

The bounded significance level tests developed are summarized in Table 2. These tests can be used to supplement the tests presented in [5] for $n \leq 9$, where the tests of [5] do not furnish a very wide variety of suitable significance levels.

8. Outline of derivations. Let us consider the range-midrange test for the more general situation in which the set of independent observations used are from arbitrary but fixed populations satisfying conditions (D). Let D_α be redefined so that the resulting test A has significance level α. Then it is easily seen that D_α is a monotone decreasing function α. Thus the significance level of the modified test A will always be less than or equal to $(1/2)^n$ if $D_\alpha > 1/2$. The significance level bounds for the tests $n = 4, \alpha = 5\%$; $n = 5, \alpha = 2.5\%$; $n = 6, \alpha = 1\%$; $n = 7, \alpha = 0.5\%$ of Table 2 were obtained from this relation and obvious significance level relations among tests A, B and C.

The significance levels (for normality) for the tests $n = 5, \alpha = 5\%$; $n = 6, \alpha = 2.5\%$; $n = 7, \alpha = 1\%$; $n = 8, \alpha = 0.5\%$ were obtained by approximate integration of the expression derived for $\Pr[(1/2 + c)x_n + (1/2 - c)x_{n-1} < \mu]$, $(0 < c < 1/2)$, for several values of c and then graphical interpolation (here α is the one-sided test significance level). The significance level bounds were determined from

$$(1/2)^n = \Pr(x_n < \mu) \leq \Pr[(1/2 + c)x_n + (1/2 - c)x_{n-1} < \mu] \leq \Pr[(1/2)(x_n + x_{n-1}) < \mu] = (1/2)^{n-1}.$$

The significance levels for the tests $n = 8, \alpha = 1\%$; $n = 9, \alpha = 0.5\%$ were obtained by considering the relations

$$\Pr\{\max [x_{n-1}, (x_n + x_{n-1})/2] < \mu\} = (1 + i)(1/2)^n, \qquad (i = 0, 1, 2, 3),$$

and applying linear interpolation to find a value c, $(0 < c < 1/2)$, such that

$\Pr\{\max\,[x_{n-1}\,,\,0.5x_n + cx_{n-2} + (1/2 - c)x_{n-1}] < \mu\}$ has the desired value. The significance level bounds were found from

$$\Pr\{(1/2)(x_n + x_{n-1}) < \mu\} \leq \Pr\{\max[x_{n-1}\,,\,0.5x_n + cx_{n-2} + (\tfrac{1}{2} - c)x_{n-1}] < \mu\}$$
$$\leq \Pr\{\max[x_{n-1}\,,\,(1/2)(x_n + x_{n-2})] < \mu\}.$$

The derivation of the power efficiencies listed in Table 2 will not be considered here. Detailed derivations can be found in [7].

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