be zero for every $\alpha > 0$ and $\beta < 0$, and thus we infer from Theorem 2 that the fundamental identity holds for all real t (if the limits a_N and b_N are chosen in accordance with the conditions of this theorem). This proposition is somewhat more general than that proved in [3] by a similar method.

It also follows from the last remark and Theorem 3 that, when $P(z = 0) < 1$, (9) can be differentiated any number of times for any real t. This proposition contains the results in [2] and [3] as special cases.

7. A generalization. We finally remark that the assumption made in Theorem 3 that the expressions containing derivatives of $\varphi_\nu(t)$ are uniformly bounded is unnecessarily restrictive. For example, it seems possible to prove that the first derivative of (2) may be obtained by differentiation under the expectation sign if the series (cf. Corollary 1 to Theorem 7.4. in [6])

$$\sum_{m=1}^{\infty} P(n = m) \sum_{\nu=1}^{m} \frac{\varphi_\nu'(t)}{\varphi_\nu(t)}$$

is uniformly convergent with respect to t.

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SPREAD OF MINIMA OF LARGE SAMPLES

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1. Theorems. Let x have the continuous cumulative distribution function $F(x)$. Let $(x_1, \cdots, x_N)$ be a sample of N independent values of x and $y = \inf (x_1, \cdots, x_N)$. Then y is a random variable with the cumulative distribution function

$$G_N(y) = 1 - (1 - F(y))^N. \tag{1}$$

Let K values of the new variable y be drawn, $(y_1, \cdots, y_K)$ and let the spread

$$w = \sup (y_1, \cdots, y_K) - \inf (y_1, \cdots, y_K).$$

Fixing K, we consider the cumulative distribution function of w, $P_N(w)$, as $N \to \infty$. That is, we have K large samples of x and wish to examine the spread among their minima. It is evident intuitively that if $F(x) = 0$ for some finite x, these minima are bounded from below and will cluster near the vanishing point of $F(x)$, making $w \to 0$ statistically as $N \to \infty$. Our theorems also show that even when $y \to -\infty$ statistically, i.e., when $F(x) = 0$ for no finite x, the spread $w \to 0$ statistically if the tail of $F(x)$ is sufficiently small (e.g. Gaussian). On the other hand, if $F(x) = 0(e^{kx})$ as $x \to -\infty$, the distribution $P_N(w)$ does not peak as $N \to \infty$, while for larger tails (e.g. algebraic) $w \to +\infty$ statistically. Two simple theorems are

I. *If*

$$\lim_{x \to -\infty} \frac{F(x)}{F(x+s)} = 1,$$

then

$$\lim_{N \to \infty} P_N(s) = 0.$$

II. *Let* $s > 0$. *If*

$$F(x_0) = 0 \textit{ for some } x_0 > -\infty, \textit{ or if}$$

$$\lim_{x \to -\infty} \frac{F(x)}{F(x+s)} = 0,$$

then

$$\lim_{N \to \infty} P_N(s) = 1.$$

Theorem I is directly applicable to distributions with algebraic tails, theorem II to Gaussian tails. We prove them both as corollaries of the more general results:

III. *If*

$$\liminf_{x \to -\infty} \frac{F(x)}{F(x+s)} = l$$

then

$$\limsup_{N \to \infty} P_N(s) \le (1 - l)^{K-1}.$$

IV. *Let* $s > 0$. If

$$F(x) = 0 \textit{ for no finite } x \textit{ and}$$

$$\limsup_{x \to -\infty} \frac{F(x)}{F(x+s)} = L,$$

then

$$\liminf_{N \to \infty} P_N(s) \ge [e^{-\alpha L} - e^{-\alpha}]^K$$

for any $\alpha > 0$.

Theorems III and IV together show that an exponential tail ($F(x) = 0(e^{kx})$) leads to a $P_N(w)$ which, asymptotically, is bounded away from 0 for any $w > 0$ and bounded away from 1 for w sufficiently small.

2. Proofs. Explicitly, for any $s \geq 0$,

$$(2) \qquad P_N(s) = K \int_{-\infty}^{\infty} [G_N(x+s) - G_N(x)]^{K-1}\, dG_N(x+s).$$

Turning now to III: given $s > 0$, choose $x_1 = x_1(\epsilon)$ so that (i) $F(x_1) \neq 0$, and (ii), $x \leq x_1$ implies

$$(3) \qquad \frac{F(x)}{F(x+s)} \geq l - \epsilon.$$

We then rewrite (2) as

$$(4) \qquad P_N(s) = \int_{-\infty}^{x_1} \left[1 - \frac{G_N(x)}{G_N(x+s)}\right]^{K-1} dG_N(x+s)^K + \int_{x_1}^{\infty}.$$

Treating $G_N(x + s)^K$ as the independent variable, the first integral may be evaluated by the mean value theorem in the form

$$(5) \qquad \left[1 - \frac{G_N(x_2)}{G_N(x_2+s)}\right]^{K-1} \int_{-\infty}^{x_1} dG_N(x+s)^K \leq \left[1 - \frac{G_N(x_N)}{G_N(x_N+s)}\right]^{K-1}$$

with an appropriate $x_2 = x_2(N)$, $-\infty \leq x_2 \leq x_1$.

Using the form (2) of the integrand in the second term of (4), we may bound the latter by

$$(6) \qquad K \int_{x_1}^{\infty} dG_N(x+s) \leq K[1 - G_N(x_1+s)],$$

since

$$G_N(x+s) - G_N(x) \leq 1.$$

Now, by factoring (1),

$$(7) \qquad \frac{G_N(x)}{G_N(x+s)} = \frac{F(x)}{F(x+s)} \frac{1 + Q + \cdots + Q^{N-1}}{1 + Q_s + \cdots + Q_s^{N-1}} \geq \frac{F(x)}{F(x+s)}$$

where $Q = 1 - F(x)$, $Q_s = 1 - F(x+s) \leq Q$. Combining (3), (4), (5), (6), and (7),

$$P_N(s) \leq [1 - l + \epsilon]^{K-1} + K[1 - G_N(x_1+s)].$$

Since $F(x_1 + s) \geq F(x_1) > 0$, we have

$$\lim_{N\to\infty} G_N(x_1+s) = 1.$$

Hence,

$$\limsup_{N\to\infty} P_N(s) \leq [1 - l + \epsilon]^{K-1}$$

and III follows by letting $\epsilon \to 0$. Then I follows immediately with $l = 1$, when we note that $P_N(s) \geq 0$.

To prove IV, choose any $\alpha > 0$. By hypothesis, for sufficiently large N we may always find $x_N = x_N(\alpha)$ such that

$$F(x_N) = \frac{\alpha L}{N}. \tag{8}$$

By hypothesis, and the monotonicity of $F(x)$, $x_N \to -\infty$ as $N \to \infty$. For any $\epsilon > 0$, therefore, we can find $N_0 = N_0(\alpha, \epsilon)$ such that $N \geq N_0$ implies

$$\frac{F(x_N)}{F(x_N + s)} \leq \frac{L}{1 - \epsilon} \tag{9}$$

or $F(x_N + s) \geq \frac{\alpha}{N}(1 - \epsilon)$. Directly from (2), since $s > 0$,

$$\begin{aligned} P_N(s) &\geq K \int_{x_N - s}^{x_N} [G_N(x + s) - G_N(x)]^{K-1} \, dG_N(x + s) \\ &\geq K \int_{x_N - s}^{x_N} [G_N(x + s) - G_N(x_N)]^{K-1} \, dG_N(x + s). \end{aligned}$$

But this last integral is of the form

$$\int K(U - G)^{K-1} \, dU = (U - G)^K,$$

whence

$$P_N(s) \geq [G_N(x_N + s) - G_N(x_N)]^K,$$

or

$$P_N(s) \geq [(1 - F(x_N))^N - (1 - F(x_N + s))^N]^K. \tag{10}$$

By (8) and (9), therefore

$$P_N(s) \geq \left[\left(1 - \frac{\alpha L}{N}\right)^N - \left(1 - \frac{\alpha(1 - \epsilon)}{N}\right)^N\right]^K$$

Since this holds for all $N \geq N_0(\alpha, \epsilon)$,

$$\liminf_{N \to \infty} P_N(s) \geq [e^{-\alpha L} - e^{-\alpha(1-\epsilon)}]^K$$

This last, in turn, now holds for any $\epsilon > 0$, hence

$$\liminf_{N \to \infty} P_N(s) \geq [e^{-\alpha L} - e^{-\alpha}]^K.$$

This now holds for any $\alpha > 0$. Maximizing on α yields a sharper bound than the result of IV. The applicable part of II follows, when $L = 0$, by letting $\alpha \to \infty$. That the conclusion of II holds when $F(x_0) = 0$ for some finite x_0 follows from (10) with x_N replaced by some x_1 such that $F(x_1) = 0$, $F(x_1 + s) > 0$.