NOTES

This section is devoted to brief research and expository articles on methology and other short items.

A GENERALIZATION OF WALD'S FUNDAMENTAL IDENTITY

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1. Summary. The fundamental identity is generalized to the case of independent random variables with non-identical distributions. The conditions for the validity of the differentiation of the identity are discussed. The results given in [1], [2], and [3] are obtained as special cases.

2. A property of cumulative sums. Let $z_1, z_2, \cdots$ be an infinite sequence of independent random variables, $F_1(z), F_2(z), \cdots$ their distribution functions (d.f.) and $\varphi_1(t), \varphi_2(t), \cdots$ their moment-generating functions so that $\varphi_\nu(t) = E(e^{tz_\nu})$. a_N and b_N are given constants ($a_N > b_N$, $N = 1, 2, \cdots$). n is defined as the smallest integer N for which $Z_N = z_1 + \cdots + z_N$ is $\geqq a_N$ or $\leqq b_N$.

We first give two lemmas.

Lemma 1. *If two positive quantities δ and ϵ can be found such that one at least of the following conditions a) and b) are satisfied*

$$a)\ P(z_\nu > \delta) > \epsilon \text{ for all } \nu \text{ and } \limsup_{N\to\infty} a_N < \infty$$

$$b)\ P(z_\nu < -\delta) > \epsilon \text{ for all } \nu \text{ and } \liminf_{N\to\infty} b_N > -\infty,$$

then for any $k \geqq 0$

$$\lim_{N\to\infty} N^k P(n > N) = 0. \tag{1}$$

An inspection of the proof of (4) in [4] shows that this formula holds when the conditions of the lemma are satisfied. The lemma follows.

Lemma 1 can be generalized as follows.

Lemma 2. *If two positive quantities δ and ϵ and a sequence $c_1, c_2, \cdots$ can be found such that one at least of the following conditions a) and b) are satisfied*

$$a)\ P(z_\nu + c_\nu > \delta) > \epsilon \text{ for all } \nu, \quad \limsup_{N\to\infty} a_N < \infty, \quad \limsup_{N\to\infty} \sum_1^N c_\nu < \infty,$$

$$b)\ P(z_\nu + c_\nu < -\delta) > \epsilon \text{ for all } \nu,$$

$$\liminf_{N\to\infty} b_N > -\infty, \quad \liminf_{N\to\infty} \sum_1^N c_\nu > -\infty,$$

then (1) is true.

PROOF: In case a) we put $z'_\nu = z_\nu + c_\nu$, $Z'_N = \Sigma z'_\nu$ and $a'_N = a_N + \Sigma_1^N c_\nu$. The inequality $Z_N \geqq a_N$ then becomes $Z'_N \geqq a'_N$. As $P(z'_\nu > \delta) > \epsilon$ and $\limsup_{N\to\infty} a'_N < \infty$, Lemma 1 can be applied to the sequence $z'_1, z'_2, \cdots$, and thus (1) is true. When conditions b) are satisfied, the proof is analogous.

3. The generalized fundamental identity. In this section we shall consider sequences of random variables of the type defined in Lemma 2. We shall prove two theorems the first of which is valid for complex values of t and the second only for real values of t.

THEOREM 1. *Assuming that*

1°. *one at least of conditions a) and b) of Lemma 2 is satisfied;*

2°. $b \leqq b_N < a_N \leqq a$, *where a and b are finite;*

3°. *for some complex (or real) value of t, $\varphi_\nu(t)$ exists for all ν and is $\neq 0$ and*

$$\liminf_{N\to\infty} | \varphi_1(t) \cdots \varphi_N(t) | > 0,$$

then

$$E[e^{tZ_n}(\varphi_1(t) \cdots \varphi_n(t))^{-1}] = 1. \tag{2}$$

PROOF. Let W_m denote the set of all sequences $z_1 \cdots z_N$ in the N-dimensional Euclidean space Ω_N for which $n = m$ $(m \leqq N)$, W'_m the projection of W_m on Ω_m and $W_{n>N}$ all sequences for which $n > N$. We have identically

$$\left[\sum_{m=1}^{N} \int_{W_m} + \int_{W_{n>N}}\right] e^{tZ_N}\, dF_1 \cdots dF_N = \int_{\Omega_N} e^{tZ_N}\, dF_1 \cdots dF_N = \varphi_1(t) \cdots \varphi_N(t).$$

Dividing by the right member and cancelling common factors we obtain

$$\sum_{m=1}^{N} (\varphi_1 \cdots \varphi_m)^{-1} \int_{W'_m} e^{tZ_m}\, dF_1 \cdots dF_m + (\varphi_1 \cdots \varphi_N)^{-1} \int_{W_{n>N}} e^{tZ_N}\, dF_1 \cdots dF_N = 1. \tag{3}$$

When $N \to \infty$ the first sum tends to the left member of (2). We thus have to investigate the last term in (3) which we denote by R_N. We can write

$$R_N = (\varphi_1 \cdots \varphi_N)^{-1} \int_{W_{n>N}} e^{tZ_N}\, dF_1 \cdots dF_N = (\varphi_1 \cdots \varphi_N)^{-1} P(n > N) E_{n>N} e^{tZ_N}. \tag{4}$$

It follows from Lemma 2 that $P(n > N) \to 0$. As $b < Z_N < a$ by 2° we conclude that $R_N \to 0$. This proves the theorem.

THEOREM 2. *If, for some real value of t, $\varphi_\nu(t)$ exists for all ν and if quantities c_ν, $\epsilon > 0$ and $\delta > 0$ can be found such that at least one of the following conditions a) and b) are satisfied for all ν*

a) $\limsup_{N\to\infty} a_N < \infty$, $\limsup_{N\to\infty} \sum_1^N c_\nu < \infty$ *and*

$$A_\nu(t, \delta) = \frac{1}{\varphi_\nu(t)} \int_{\delta - c_\nu}^{\infty} e^{tz}\, dF_\nu(z) > \epsilon, \qquad (\nu = 1, 2, \cdots), \tag{5a}$$

b) $\liminf_{N\to\infty} b_N > -\infty$, $\liminf_{N\to\infty} \sum_1^N c_\nu > -\infty$ *and*

$$B_\nu(t, \delta) = \frac{1}{\varphi_\nu(t)} \int_{-\infty}^{-\delta-c_\nu} e^{tz}\, dF_\nu(z) > \epsilon, \qquad (\nu = 1, 2, \cdots), \tag{5b}$$

then (2) *holds.*

The conditions of the theorem become more attractive if the theorem is limited to the somewhat less general cases mentioned in the Corollary below. The above formulation has been chosen mainly because of an important application to identical variables in Sec. 6.

PROOF. The theorem is proved if we can show that R_N in (4) tends to zero when $N \to \infty$. For that purpose we use the transformation (cf [5] and [3])

$$G_\nu(z; t) = \frac{1}{\varphi_\nu(t)} \int_{-\infty}^{z} e^{tz}\, dF_\nu(z), \qquad (\nu = 1, 2, \cdots). \tag{6}$$

$G_\nu(z; t)$ is obviously a d.f. for every real t (for which $\varphi_\nu(t)$ exists). When (5a) holds,

$$P[z_\nu + c_\nu > \delta \mid G_\nu(z; t)] = A(t, \delta).$$

Here the expression in the left member denotes the probability that $z_\nu + c_\nu > \delta$, when G_ν is the d.f. of z_ν.

Consequently, when conditions a) are fulfilled, a sequence of random variables with the d.f:s $G_1(z; t)$, $G_2(z; t)$, $\cdots$ or, with one notation, $G(t)$ satisfies the conditions a) of Lemma 2. It follows that

$$\lim_{N\to\infty} P(n > N \mid G(t)) = 0.$$

Introducing $G_\nu(z; t)$ in R_N we find

$$R_N = \int_{W_n > N} dG_1 \cdots dG_N = P(n > N \mid G(t)).$$

Consequently $R_N \to 0$. When conditions b) are fulfilled, the proof is analogous.

COROLLARY TO THEOREM 2. *If* 1° $\varphi_\nu(t)e^{tc_\nu} \leqq H(t) < \infty$, 2° t *is positive and conditions* a) *of Lemma* 2 *hold or* t *is negative and conditions* b) *of Lemma* 2 *hold, then the generalized fundamental identity is true.*

For, in the first case

$$A_\nu(t, \delta) \geqq \frac{e^{t(\delta-c_\nu)}}{\varphi_\nu(t)} \int_{\delta-c_\nu}^{\infty} dF_\nu \geqq \frac{\epsilon e^{t\delta}}{H(t)} = \epsilon_1(t)$$

so that (5a) is satisfied, and similarly when t is negative.

The following special case deserves particular attention as it covers most cases occurring in practice and the conditions become very simple: *If a sequence of random variables satisfies conditions* a) *and* b) *of Lemma* 1 *simultaneously, a sufficient condition for the validity of* (2) *for some given real value of* t *is that the sequence* $\varphi_\nu(t)$ *is bounded.*

4. Application to Poisson variables. As an application of (2) we consider a sequence of Poisson variables with the parameters λm_ν, where λ is a positive quantity and m_ν are positive integers. From the well-known formula

$$\varphi_\nu(t) = e^{\lambda m_\nu (e^t - 1)}$$

we easily conclude that the conditions of Theorem 1 are valid if $R(e^t) \geqq 1$. (With $\delta < 1$ in (5a) we find that (2) holds even for negative t.) If, in particular, we choose t so that $e^t = 1 + \frac{2\pi i k}{\lambda} = c_k$, we have the simple formula

$$E(c_k^{Z_n}) = 1, \qquad (k = 1, 2, \cdots).$$

5. Differentiation of the generalized fundamental identity. In this section t is assumed to be real. We denote the kth derivative of $\varphi_\nu(t)$ by $\varphi_\nu^{(k)}(t)$. We shall prove the following theorem which corresponds to Theorems 1 and 2.

THEOREM 3. *If for all t in a closed interval I the conditions stated in Theorems 1 or 2 are satisfied and if, in addition, the functions $\left| \frac{\varphi_\nu^{(k)}(t)}{\varphi_\nu(t)} \right|$ are uniformly bounded with respect to both ν and t (in I) for $k = 1, 2, \cdots r$, then the generalized fundamental identity may be differentiated r times with respect to t for any t in the interior of I.*

We use a method of proof which is similar to that used in [2]. We first show that the sum in (3) may be differentiated r times under the integral signs and secondly that the rth derivative of R_N tends to zero uniformly in t when $N \to \infty$.

The rth derivative of the general term of the series in (3) consists of a finite number of terms of the form

$$J_m(t) = (\varphi_1 \cdots \varphi_m)^{-1} H_\mu \int_{W_m'} Z_m^\lambda e^{tZ_m} \, dF_1 \cdots dF_m \qquad (\mu \leqq \lambda; \mu, \lambda = 1, 2, \cdots r),$$

and the rth derivative of R_N in (4) consists of a finite number (which does not depend on N) of similar expressions with N substituted for m and $W_{n>N}$ for W_m'. H_μ is a sum of m^μ and N^μ terms respectively which is symmetric in ν. The terms are functions of $\frac{\varphi_\nu^{(k)}(t)}{\varphi_\nu(t)}$ $(k \leqq \lambda; \nu = 1, 2, \cdots m)$ and are thus majorated by the same constant C.

Further, we can always find a positive quantity t_0 such that for all t in I

$$| Z_m^\lambda e^{tZ_m} | \leqq e^{t_0 |Z_m|} \leqq (e^{t_0 Z_m} + e^{-t_0 Z_m}).$$

Hence

$$(7) \qquad | J_m(t) | \leqq (\varphi_1 \cdots \varphi_m)^{-1} C m^\mu \int_{W_m'} (e^{t_0 Z_m} + e^{-t_0 Z_m}) \, dF_1 \cdots dF_m .$$

The rest of the proof is divided into two parts corresponding to the conditions of Theorem 1 and those of Theorem 2.

When the conditions of Theorem 2 are fulfilled we make the transformation (6) in (7) with $t = t_0$ and $t = -t_0$. Then

$$| J_m(t) | \leqq Cm^{\mu}[P(n = m \mid G(t_0)) + P(n = m \mid G(-t_0))] \leqq 2Cm^{\mu} < \infty.$$

This justifies the differentiation of the series in (3).

Substituting N for m and $n > N$ for $n = m$ in the above expression we further have

$$| J_N(t) | \leqq CN^{\mu}[P(n > N \mid G(t_0)) + P(n > N \mid G(-t_0))],$$

and conclude from Lemma 2 with $k = \mu$ in (1) that $J_N(t)$ tends to zero uniformly in t. It follows that the rth derivative of R_N also tends to zero uniformly in t.

In the second part of the proof we assume the conditions of Theorem 1 to be satisfied. We then write (7) in the following form

$$(8) \qquad | J_m(t) | \leqq C(\varphi_1 \cdots \varphi_m)^{-1} m^{\mu} P(n = m) E_{n=m}(e^{t_0 Z_m} + e^{-t_0 Z_m}),$$

where $E_{n=m}$ signifies the conditional expectation when it is known that $n = m$. From the definition of n it follows that, when $n = m$, we have $b_{m-1} < Z_{m-1} < a_{m-1}$ and $Z_m \geqq a_m$ or $\leqq b_m$. Hence

$$E_{n=m}(e^{t_0 Z_m}) \leqq E_{n=m}(e^{t_0 Z_m} \mid Z_m \geqq a_m) = E_{n=m}[e^{t_0(Z_{m-1}+z_m)} \mid Z_{m-1} + z_m \geqq a_m]$$
$$\leqq e^{t_0 a_{m-1}} E[e^{t_0 z_m} \mid z_m > a_m - b_{m-1}] < \infty.$$

The second exponential can be treated in a similar way. Thus $J_m(t)$ is majorated by a finite expression.

Finally, we substitute N for m and $n > N$ for $n = m$ in (8). I being a closed interval it follows from condition 3° in Theorem 1 that we can find a constant C such that

$$| J_N(t) | \leqq CN^{\mu} P(n > N) E_{n>N}(e^{t_0 Z_N} + e^{-t_0 Z_N}).$$

From the definition of n and condition 2° in Theorem 1 we have $b < Z_N < a$. An application of Lemma 2 then shows that $J_N(t)$ tends to zero uniformly in t. This proves the theorem.

COROLLARY TO THEOREM 3. *When the conditions stated in Corollary of Theorem 2 are fulfilled for all t in the closed interval I, Theorem 3 is true.*

This is obvious.

6. The fundamental identity for identically distributed variables. In the special case of identically distributed variables for which $P(z = 0) < 1$ and $0 < \varphi(t) < \infty$ we infer from Theorem 1 that the fundamental identity

$$(9) \qquad E[e^{tZ_n}(\varphi(t))^{-n}] = 1$$

holds if t is complex and $| \varphi(t) | \geqq 1$. This is the case discussed in [1].

Further, when $P(z = 0) < 1$, the integrals $\int_{\alpha}^{\infty} e^{tz} dF$ and $\int_{-\infty}^{\beta} e^{tz} dF$ cannot both be zero for every $\alpha > 0$ and $\beta < 0$, and thus we infer from Theorem 2 that the fundamental identity holds for all real t (if the limits a_N and b_N are chosen in accordance with the conditions of this theorem). This proposition is somewhat more general than that proved in [3] by a similar method.

It also follows from the last remark and Theorem 3 that, when $P(z = 0) < 1$, (9) can be differentiated any number of times for any real t. This proposition contains the results in [2] and [3] as special cases.

7. A generalization. We finally remark that the assumption made in Theorem 3 that the expressions containing derivatives of $\varphi_\nu(t)$ are uniformly bounded is unnecessarily restrictive. For example, it seems possible to prove that the first derivative of (2) may be obtained by differentiation under the expectation sign if the series (cf. Corollary 1 to Theorem 7.4. in [6])

$$\sum_{m=1}^{\infty} P(n = m) \sum_{\nu=1}^{m} \frac{\varphi_\nu'(t)}{\varphi_\nu(t)}$$

is uniformly convergent with respect to t.

REFERENCES

[1] A. WALD, "On cumulative sums of random variables," *Annals of Math. Stat.*, Vol. 15 (1944), p. 285.

[2] A. WALD, "Differentiation under the expectation sign in the fundamental identity of sequential analysis," *Annals of Math. Stat.*, Vol. 17 (1946), pp. 493–497.

[3] G. E. ALBERT, "A note on the fundamental identity of sequential analysis," *Annals of Math. Stat.*, Vol. 18 (1947), pp. 593–596 and Vol. 19 (1948), pp. 426–427.

[4] C. STEIN, "A note on cumulative sums," *Annals of Math. Stat.*, Vol. 17 (1946), pp. 498–499.

[5] H. CRAMÉR, Sur un nouveau theorème-limite de la théorie des probabilités, *Actualités, scientifiques et industrielles*, no. 736, Hermann et Cie., 1938, p. 5.

[6] J. WOLFOWITZ, "The efficiency of sequential estimates and Wald's equation for sequential processes," *Annals of Math. Stat.*, Vol. 18 (1947), pp. 228–229.

SPREAD OF MINIMA OF LARGE SAMPLES

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1. Theorems. Let x have the continuous cumulative distribution function $F(x)$. Let $(x_1, \cdots, x_N)$ be a sample of N independent values of x and $y = \inf (x_1, \cdots, x_N)$. Then y is a random variable with the cumulative distribution function

$$G_N(y) = 1 - (1 - F(y))^N. \tag{1}$$

Let K values of the new variable y be drawn, $(y_1, \cdots, y_K)$ and let the spread

$$w = \sup (y_1, \cdots, y_K) - \inf (y_1, \cdots, y_K).$$