PROBLEMS IN PLANE SAMPLING

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1. Summary. After consideration of the relative accuracies of systematic and stratified random sampling in one dimension the problem of estimation of linear sampling error is discussed.

Methods of sampling an area are proposed, and expressions for the accuracies of these methods are derived. These expressions are compared for large samples, with special reference to correlation functions which appear to be theoretically and practically justified, and systematic sampling is found to be more accurate than stratified random sampling in many cases. Methods of estimating sampling errors are again considered, and examples given. The paper concludes with some remarks on the problem of trend in the population sampled.

2. Accuracy of systematic and stratified random samples in one dimension. W. G. Cochran [1] has given expressions to the variances of the means of samples of size n drawn from a population $x_1 x_2 \cdots x_{nk}$ when the method of sampling is random (r), stratified random (st) and systematic (sy). He assumes the elements $x_1 x_2 \cdots x_{nk}$ to be drawn from a population in which

$$E(x_i) = \mu, \qquad E(x_i - \mu)^2 = \sigma^2, \qquad E(x_i - \mu)(x_{i+u} - \mu) = \rho_u \sigma^2$$

where $\rho_u \geq \rho_v \geq 0$ whenever $u < v$, and derives the expressions

$$(1) \quad \sigma_r^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right)\left[1 - \frac{2}{kn(kn-1)}\sum_{u=1}^{kn-1}(kn-u)\rho_u\right]$$

$$(2) \quad \sigma_{st}^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right)\left[1 - \frac{2}{k(k-1)}\sum_{u=1}^{k-1}(k-u)\rho_u\right]$$

$$(3) \quad \sigma_{sy}^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right)$$

$$\cdot\left[1 - \frac{2}{kn(k-1)}\sum_{u=1}^{kn-1}(kn-u)\rho_u + \frac{2k}{n(k-1)}\sum_{u=1}^{n-1}(n-u)\rho_{ku}\right].$$

Using these expressions which are linear functions of the ρ_u Cochran compares the relative efficiencies of the methods of sampling for several types of correlogram. It is worth noting that (1), (2) and (3) can be derived under more general conditions than Cochran considered. If we assume that (a) each x_i is a sample from a population with mean μ_i and variance σ_i^2, (b) that μ_i is distributed about mean μ with variance σ^2, (c) that $E(\mu_i - \mu)(\mu_j - \mu) = \rho_{ij}\sigma^2$, and (d) that $\rho_u = \frac{1}{kn-u}\sum_{i=1}^{kn-u}\rho_{i,i+u}$, then it is not difficult to show that (1), (2) and (3)

require the addition of a superposed variation $\frac{1}{n}\left(1 - \frac{1}{k}\right) \cdot \frac{1}{kn} \sum_{i=1}^{kn} \sigma_i^2$ to the right-hand side of the equations. Thus it should be remembered that Cochran's results give theoretical maxima to the relative efficiencies of the various methods of sampling, while ρ_u is the mean correlation between samples u apart. This result is perhaps interesting in connection with sampling for say, insect infestation, when at each point there will be a mean level of infestation and the sample will be distributed in a Poisson distribution about this mean. Then the superposed variation is

$$\frac{1}{n}\left(1 - \frac{1}{k}\right) \cdot \frac{1}{kn} \sum_{i=1}^{kn} \mu_i \sim \frac{1}{n}\left(1 - \frac{1}{k}\right)\mu.$$

If we are sampling a continuous process[1], for n large we can write down the integral equivalents of (1), (2) and (3)

$$\sigma_r^2 \sim \frac{\sigma^2}{n}$$

(4) $$\sigma_{st}^2 \sim \frac{\sigma^2}{n}\left[1 - \frac{2}{d^2}\int_0^d (d - u)\rho_u \,\delta u\right]$$

(5) $$\sigma_{sy}^2 \sim \frac{\sigma^2}{n}\left[1 - \frac{2}{d}\int_0^\infty \rho_u \,\delta u + 2\sum_{u=1}^\infty \rho_{du}\right]$$

where ρ_u is the mean correlation between successive elements of the sample, u apart and d is the mean distance between samples. We have thus

$$\frac{\sigma_{st}^2 - \sigma_{sy}^2}{\sigma_r^2} \sim \frac{2}{d}\left[\int_0^d \frac{u}{d}\rho_u \,\delta u + \int_d^\infty \rho_u \,\delta u - d\sum_{u=1}^\infty \rho_{du}\right],$$

which can often be used to investigate, quickly and roughly, with the aid of a graph the difference between the efficiencies of stratified random and systematic sampling. Figure 1 shows how this is done for four types of correlogram.

For a continuous Markoff scheme, we have $\rho_u = \rho^u$ and

$$\sigma_{st}^2 \sim \frac{\sigma^2}{n}\left[1 + \frac{2}{\log \rho^d} + \frac{2}{(\log \rho^d)^2} - \frac{2\rho^d}{(\log \rho^d)^2}\right],$$

$$\sigma_{sy}^2 \sim \frac{\sigma^2}{n}\left[1 + \frac{2}{\log \rho^d} + \frac{2\rho^d}{1 - \rho^d}\right],$$

which agree with Cochran's results.

3. Replication and the estimation or error. Yates [2] has pointed out the difficulties attached to the estimation of error for a systematic sample. It will, however, be worthwhile to investigate this point using the above formulae.

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[1] In practice we can sample a continuous process only as if it were a discontinuous process with k large.

For random, stratified random and systematic sampling, if n is large and k is regarded as constant, then the variance of the estimate of the mean will be of the form $\sigma^2 F(k)/n$, where $F(k)$ is virtually independent of n. Thus, if we have any method which provides an estimate of error for the samples it will be possible to split the series to be sampled into several equal parts (or blocks) to obtain an estimate of error of the mean of each part and to combine these to obtain a more accurate estimate of the error of the overall mean. In fact, if n is very large, we may wish to reduce our number of observations by obtaining estimates of error from a random selection of these parts. For stratified random sampling, $F(k)$ is completely independent of n, so that we may combine our estimates of error from each strata. This leads us to the commonly used method of taking q randomly chosen elements per strata, and combining the sets of variances of $q - 1$ degrees of freedom to form an estimate of error. If we make our samples exclusive, i.e. no two elements can coincide, then this variance has to be multiplied by $1 - q/k$ to give the estimated variance of the sample mean.

We can in the same way estimate the variance of the mean of a systematic sample by using sets of q systematic samples of sufficient length with randomly-chosen starting points. This sampling will, however, be more difficult to carry out in practice, and we might consider other methods. Our systematic samples may be chosen to be invariable in each part or block into which the series is split so that our sampling procedure involves, in all, only q systematic samples, or we might follow the method advocated by Yates of choosing our q samples to be evenly spaced, so that they are subsamples of a larger systematic sample. Whereas this latter method has simplicity and its possible incorporation into a more extensive scheme to recommend it, its use has to be very carefully considered. If we consider the discrete case, we wish to estimate

$$(6) \qquad \sigma^2 \left(1 - \frac{2}{k-1} \sum_{u=1}^{\infty} \rho_u + \frac{2k}{k-1} \sum_{u=1}^{\infty} \rho_{ku} \right),$$

but any estimate of variance based on q evenly-spaced systematic samples can contain only terms of the form $\rho_{ku/q}$, and while an estimate of variance based on q randomly-chosen systematic samples will obviously be limited, it will, in most cases, be more representative. As an example, suppose we take $k = 16$ and $q = 4$ then we can compare the relative occurrences of observing the correlations $\rho_1 \cdots \rho_{15}$ in the estimate of variance. Six examples of this are given in table 1, the random numbers having been drawn from Fisher and Yates tables; ρ_u and ρ_{16-u} being shown together, since they occur equally frequently. The table demonstrates how randomly-chosen samples, even as nearly systematic as the first two randomly-chosen samples will avoid systematically sampling the correlogram. It is obvious that in most cases either method will be fairly good but the use of this latter will usually be the more accurate. Comparisons are made in table 2 for various types of correlogram using the samples indicated in table 1. It is, of course, possible to postulate theoretically many kinds of

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[2] Throughout this paper δ is used for the differential sign to prevent confusion with d.

correlogram for which the equal-spaced sets of systematic samples will break down, but ultimately we must decide with reference to the types of correlogram

TABLE 1

Frequency of occurrence of the serial correlations ρ_1, ρ_2 ... ρ_{15} in the estimate of variance when 4 systematic samples each with spacing 16 units are taken

ρ	4 evenly-spaced systematic samples	4 systematic samples with random starting points at						Total frequencies
		4, 7, 8, 12,	3, 7, 8, 12,	3, 6, 10, 13,	4, 6, 7, 14,	2, 8, 11, 15,	2, 6, 11, 16	
1, 15		1	1		1			3
2, 14					1		1	2
3, 13		1		2	1	2		6
4, 12	4	2	2	1		1	1	7
5, 11		1	2				2	5
6, 10				1	1	1	1	4
7, 9			1	2	1	2	1	7
8	4	2			2			4

TABLE 2

Values of $\frac{2}{15} \sum\limits_{u=1}^{15} \rho_u$ as estimated by systematic samples

ρ_u	Evenly-spaced systematic samples	Systematic samples with random starting points						Mean	Expected
		1	2	3	4	5	6		
$1-0.2\,u$, $(u = 1, .5)$	0.17	0.27	0.20	0.17	0.30	0.17	0.13	0.21	0.27
$1-0.1\,u$, $(u = 1, .10)$	0.53	0.62	0.58	0.53	0.60	0.53	0.53	0.57	0.60
2^{-u}	0.04	0.13	0.12	0.06	0.15	0.06	0.07	0.10	0.13
$2^{-u/4}$	0.58	0.66	0.64	0.60	0.66	0.60	0.60	0.63	0.65
Kendall's Series 1	−0.14	0.03	0.00	−0.05	0.16	−0.05	−0.05	0.01	0.07

* Naturally the use of this method of estimating the sampling error assumes that the correlation between the corresponding elements in each part or block into which the series is split may be neglected, i.e. in this case that the terms ρ_{16} and above are negligible. In this case $\rho_{16} = 1/16$ and consequently the term $2(\frac{1}{15}\sum\limits_{u=1}^{\infty}\rho_u - \frac{16}{15}\sum\limits_{u=1}^{\infty}\rho_{16u}) = 0.56$, required in

(6) differs slightly from the term $\frac{2}{15}\sum\limits_{u=1}^{15}\rho_u = 0.65$ which we are attempting to estimate.

experienced. We shall consider this point further, after we have dealt with two-dimensional sampling.

4. Methods of sampling in 2 dimensions.

The number of ways in which we can sample a two-dimensional space[3] is large, since we can employ random,

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[3] We shall, in general, consider our two-dimensional space to be rectangular, but it is not difficult to draw similar conclusions for an area of any shape.

stratified random or systematic sampling in either direction. Thus we will be able to consider every possible combination of these methods, e.g. random in
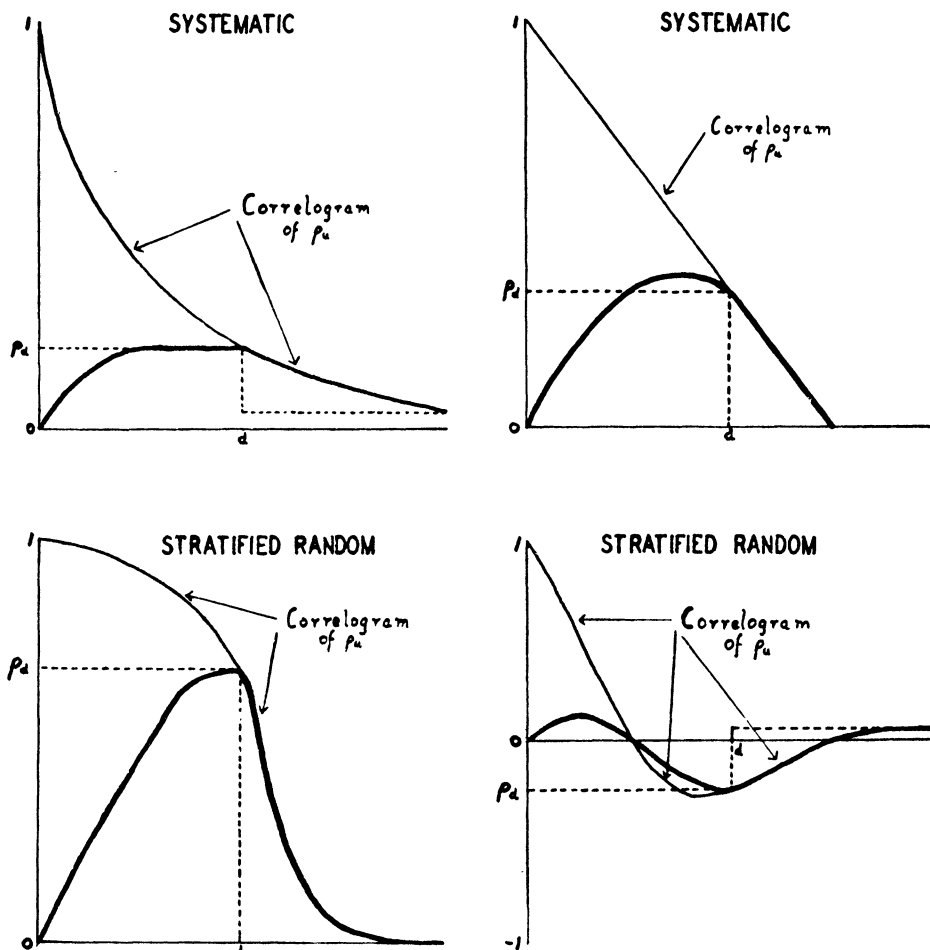

FIG. 1. Graphical comparison of the efficiencies of systematic and stratified random sampling for various correlation functions. The thick line gives the function

$$f_1(u) = u\rho_u/d, \qquad\qquad 0 \leq u \leq d$$
$$\qquad\quad = \rho_u, \qquad\qquad\qquad d \leq u,$$

and the dotted line the function

$$f_2(u) = \rho_{id}, \qquad\qquad (i-1)d < u \leq id.$$

Thus systematic sampling is more or less efficient than stratified random sampling according to whether the area under the thick line is greater or less than the area under the dotted line. The most efficient method is indicated on each graph.

one direction and systematic in another will be denoted by $r \cdot sy$. Furthermore we can consider the sets of samples in one direction to be aligned with one another, or to be independently determined. The suffix 1 will be used to denote
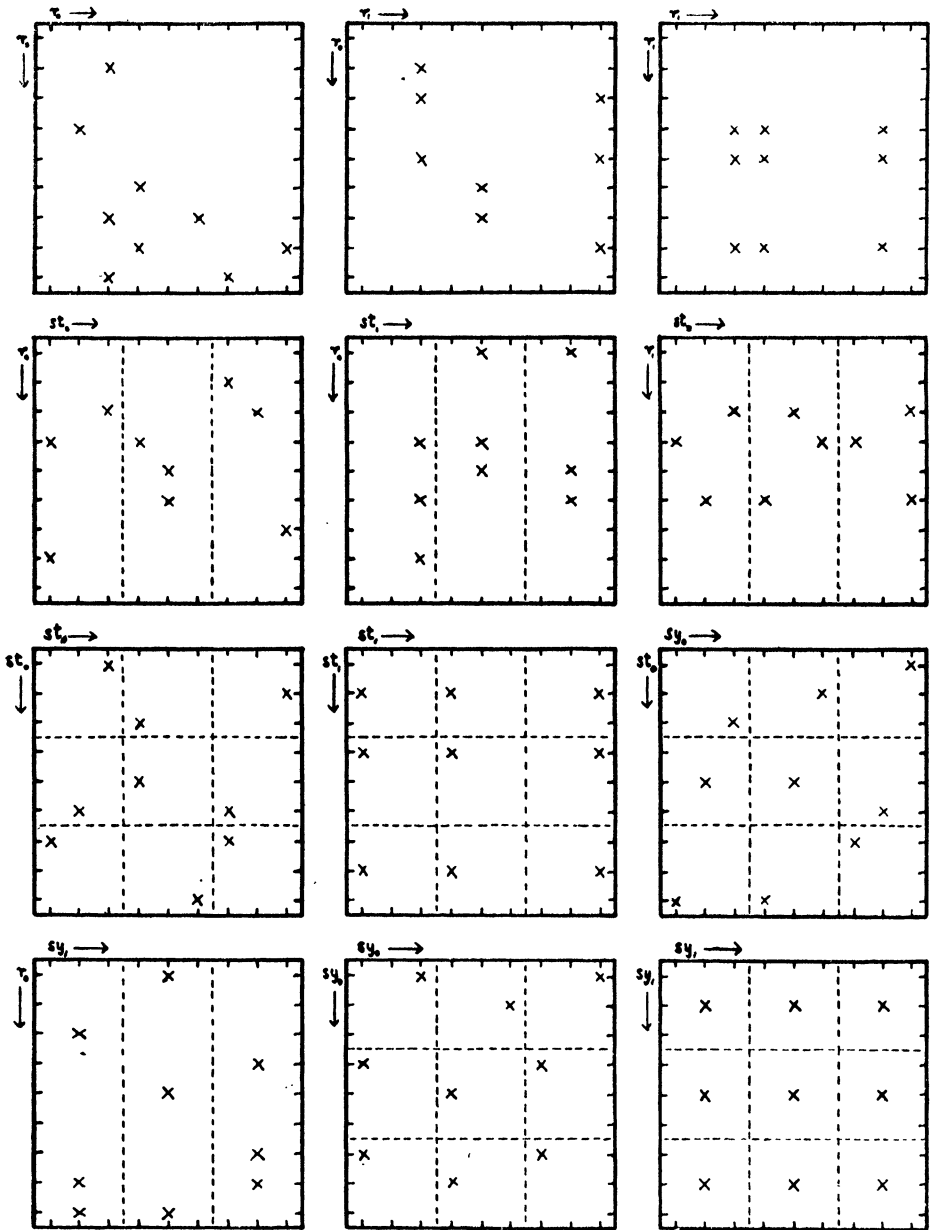
FIG. 2. Methods of sampling a field. In this case, $n_1 = n_2 = k_1 = k_2 = 3$.

aligned samples while suffix 0 will denote independent samples, e.g. we might sample according to the system $r_1 s y_0$. Examples of several methods of sampling are given in Figure 2.

5. Accuracy of sampling in two dimensions. Suppose we consider a sample of $n_1 n_2$ elements drawn from the elements $x_{ij}(i = 1, 2, \cdots n_1 k_1, j = 1, 2, \cdots n_2 k)$, (which form a single finite population drawn from an infinite hypothetical population), such that the mean spacing in the two directions is k_1 and k_2. These parameters will, if necessary, be indicated in brackets after the method of sampling, e.g. $r_1 s y_0(n_1 k_1 ; n_2 k_2)$.

Let X denote the mean of a sample formed by the method considered, and x' a member of this sample. Suppose, also, that the x_{ij} are drawn from a population in which

$$E(x_{ij}) = \mu, \qquad E(x_{ij} - \mu)^2 = \sigma^2,$$

$$E(x_{ij} - \mu)(x_{i+u,j+v} - \mu) = \rho_{ijuv}\sigma^2,$$

Further we may average ρ_{ijuv} over all possible values of i and j to define $\rho_{uv} = \rho_{-u,-v}$ by the relation

$$\sum_i \sum_j \rho_{ijuv} = (k_1 n_1 - |u|)(k_2 n_2 - |v|)\rho_{uv}.$$

The purpose of these definitions is to allow to eliminate the difficulties associated with the parameters of finite populations by considering this population as being itself a sample from an infinite population. Cochran employs a similar device.

5a. *Random sampling.* It is not difficult to see that

$$\sigma^2(X) = \tfrac{1}{2} E(X_1 - X_2)^2 = E(X_1 - \mu)^2 - E(X_1 - \mu)(X_2 - \mu),$$

where X_1 and X_2 are independent samples.
Also

$$E(X_1 - \mu)(X_2 - \mu) = E(x_1' - \mu)(x_2' - \mu)$$

$$= \frac{\sigma^2}{k_1 k_2 n_1 n_2}\left[1 + \frac{1}{k_1 k_2 n_1 n_2}\sum\sum (k_1 n_1 - |u|)(k_2 n_2 - |v|)\rho_{uv}\right]$$

where the double summation[4] exists over the region S given by $|u| \leq k_1 n_1$, $|v| \leq k_2 n_2$ and excludes $u = v = 0$. We thus have to evaluate $E(X_1 - \mu)^2$ for the different types of random sampling.

It is easily shown that

$$E(X_1 - \mu)^2 = \frac{\sigma^2}{n_1 n_2}$$

$$\cdot \left[1 + \frac{n_1 n_2 - 1}{k_1 k_2 n_1 n_2(k_1 k_2 n_1 n_2 - 1)}\sum\sum (k_1 n_1 - |u|)(k_2 n_2 - |v|)\rho_{uv}\right]$$

$$\text{for} \quad r_0 r_v,$$

$$= \frac{\sigma^2}{n_1 n_2}\left[1 + \frac{n_1 - 1}{k_1 k_2 n_1(k_1 k_2 n_1 n_2 - 1)}\sum\sum (k_1 n_1 - |u|)(k_2 n_2 - |v|)\rho_{uv}\right.$$

$$\left. + \frac{2(n_2 - 1)}{k_2 n_2(k_2 n_2 - 1)}\sum_{v=1}^{k_2 n_2} (k_2 n_2 - v)\rho_{0v}\right]$$

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[4] In general, unless otherwise stated, double summations will exist over the region for which the coefficients are positive, excluding $u = v = 0$.

$$\text{for} \quad r_1 r_0,$$

$$= \frac{\sigma^2}{n_1 n_2} \left[1 + \frac{(n_1 - 1)(n_2 - 1)}{k_1 k_2 n_1 n_2 (k_1 k_2 n_1 n_2 - 1)} \sum \sum (k_1 n_1 - |u|)(k_2 n_2 - |v|)\rho_{uv} \right.$$

$$\left. + \frac{2(n_2 - 1)}{k_2 n_2 (k_2 n_2 - 1)} \sum_{v=1}^{k_2 n_2} (k_2 n_2 - v)\rho_{0v} + \frac{2(n_1 - 1)}{k_1 n_1 (k_1 n_1 - 1)} \sum_{u=1}^{k_1 n_1} (k_1 n_1 - u)\rho_{u0} \right]$$

$$\text{for} \quad r_1 r_1,$$

whence

$$(7) \qquad \begin{aligned} \sigma^2(r_0 r_0) &= \frac{1}{n_1 n_2} \left(1 - \frac{1}{k_1 k_2} \right) \sigma^2 \\ &\cdot \left[1 - \frac{1}{k_1 k_2 n_1 n_2 (k_1 k_2 n_1 n_2 - 1)} \sum \sum (k_1 n_1 - |u|)(k_2 n_2 - |v|)\rho_{uv} \right] \end{aligned}$$

$$(8) \qquad \begin{aligned} \sigma^2(r_1 r_0) &= \frac{1}{n_1 n_2} \left(1 - \frac{1}{k_1 k_2} \right) \sigma^2 \left[1 - \frac{k_1 k_2 n_2 - 1}{(k_1 k_2 - 1) k_1 k_2 n_1 n_2 (k_1 k_2 n_1 n_2 - 1)} \right. \\ &\cdot \sum \sum (k_1 n_1 - |u|)(k_2 n_2 - |v|)\rho_{uv} \\ &\left. + \frac{2(n_2 - 1)}{k_2 n_2 (k_2 n_2 - 1)} \sum_{1}^{k_2 n_2} (k_2 n_2 - v)\rho_{0v} \right] \end{aligned}$$

$$(9) \qquad \begin{aligned} \sigma^2(r_1 r_1) &= \frac{1}{n_1 n_2} \left(1 - \frac{1}{k_1 k_2} \right) \sigma^2 \left[1 - \frac{k_1 k_2 (n_1 + n_2 - 1) - 1}{(k_1 k_2 - 1) k_1 k_2 n_1 n_2 (k_1 k_2 n_1 n_2 - 1)} \right. \\ &\cdot \sum \sum (k_1 n_1 - |u|)(k_2 n_2 - |v|)\rho_{uv} + \frac{2k_1 (n_2 - 1)}{(k_1 k_2 - 1) n_2 (k_2 n_2 - 1)} \\ &\left. \cdot \sum_{v=1}^{k_2 n_2} (k_2 n_2 - v)\rho_{0v} + \frac{2k_2 (n_1 - 1)}{(k_1 k_2 - 1) n_1 (k_1 n_1 - 1)} \sum_{u=1}^{k_1 n_1} (k_1 n_1 - u)\rho_{u0} \right]. \end{aligned}$$

5b. *Stratified random sampling.* We can deduce the variances for some methods of taking stratified samples if $\bar{x}_i'$, the mean of the elements sampled in the ith stratum, is independent of $\bar{x}_j'$, since we will then have

$$E(X - \bar{x})^2 = E(\bar{x}_i' - \bar{x}_i)^2 / n,$$

where $\bar{x}$ is the mean of the finite population which is sampled. Hence

$$(10) \qquad \begin{aligned} \sigma^2(st_0 \, r_0) &= \frac{1}{n_1} \sigma^2 \{ r_0 \, r_0(1, \, k_1 \, ; \, n_2 \, k_2) \} \\ &= \frac{1}{n_1 n_2} \left(1 - \frac{1}{k_1 k_2} \right) \sigma^2 \left[1 - \frac{1}{k_1 k_2 n_2 (k_1 k_2 n_2 - 1)} \right. \\ &\left. \cdot \sum \sum (k_1 - |u|)(k_2 n_2 - |v|)\rho_{uv} \right], \end{aligned}$$

$$\sigma^2(st_1\, r_0) = \frac{1}{n_1\, n_2}\left(1 - \frac{1}{k_1\, k_2}\right)\sigma^2\left[1 - \frac{1}{k_1\, k_2\, n_2(k_1\, k_2 - 1)}\right.$$

(11)
$$\cdot \sum\sum (k_1 - |\, u\, |)(k_2\, n_2 - |\, v\, |)\rho_{uv}$$

$$\left. + \frac{2k_1(n_2 - 1)}{(k_1\, k_2 - 1)n_2(k_2\, n_2 - 1)}\sum_{v=1}^{k_2 n_2}(k_2\, n_2 - v)\rho_{uv}\right],$$

$$\sigma^2(st_0\, st_0) = \frac{1}{n_1\, n_2}\left(1 - \frac{1}{k_1\, k_2}\right)\sigma^2\left[1 - \frac{1}{k_1\, k_2(k_1\, k_2 - 1)}\right.$$

(12)
$$\left. \cdot \sum\sum (k_1 - |\, u\, |)(k_2 - |\, v\, |)\rho_{uv}\right].$$

To estimate the variance of other methods of sampling, we will make use of a general formula which we might have used to derive the expressions (8)–(12).

If x_i' is any element of the sample X, then

$$(X - \bar{x})^2 = \frac{1}{n_1\, n_2}\left[\sum (x_i' - \bar{x})^2 - \sum (x_i' - X)^2\right]$$

$$= \frac{1}{n_1\, n_2}\left[\sum (x_i' - \bar{x})^2 - \frac{n_1\, n_2 - 1}{n_1\, n_2}\sum (x_i' - \mu)^2\right.$$

$$\left. + \frac{2}{n_1\, n_2}\sum\sum (x_i' - \mu)(x_j' - \mu)^\delta\right],$$

whence

$$\sigma^2(X) = E(X - \bar{x})^2$$

$$= \frac{k_1\, k_2\, n_1\, n_2 - 1}{k_1\, k_2\, n_1\, n_2}\sigma^2\left[1 - \frac{1}{k_1\, k_2\, n_1\, n_2(k_1\, k_2\, n_1\, n_2 - 1)}\sum\sum (k_1\, n_1 - |\, u\, |)\right.$$

(13)
$$\left. \cdot (k_2\, n_2 - |\, v\, |)\rho_{uv}\right] - \frac{n_1\, n_2 - 1}{n_1\, n_2}\sigma^2 + \frac{n_1\, n_2 - 1}{n_1\, n_2}E(x_i' - \mu)(x_j' - \mu)$$

$$= \frac{1}{n_1\, n_2}\left(1 - \frac{1}{k_1\, k_2}\right)\sigma^2\left[1 - \frac{1}{k_1\, k_2\, n_1\, n_2(k_1\, k_2 - 1)}\sum\sum (k_1\, n_1 - |\, u\, |)\right.$$

$$\left. \cdot (k_2\, n_2 - |\, v\, |)\rho_{uv} + \frac{k_1\, k_2(n_1\, n_2 - 1)}{k_1\, k_2 - 1}\frac{E(x_i' - \mu)(x_j' - \mu)}{\sigma^2}\right].$$

Thus, provided that we can estimate $E(x_i' - \mu)(x_j' - \mu)/\sigma^2$ the expression (13) gives the error for all methods of sampling.

As an example, we might deduce the expression (12). If we choose any member x_i', then a second member x_j' will be located at random with respect to x_i' except that there will be $k_1 k_2 - 1$ positions in the same stratum as x_i' that x_j' will not be able to occupy. Thus the expected correlation $E(x_i' - \mu)(x_j' - \mu)/\sigma^2$ will be given by

(14)
$$\frac{1}{k_1^2\, k_2^2\, n_1\, n_2(n_1\, n_2 - 1)}\sum\sum (k_1\, n_1 - |\, u\, |)(k_2\, n_2 - |\, v\, |)\rho_{uv}$$

$$- \frac{1}{k_1^2\, k_2^2(n_1\, n_2 - 1)}\sum\sum (k_1 - |\, u\, |)(k_2 - |\, v\, |)\rho_{uv}.$$

If we substitute (14) into (13), we will obtain expression (12) for the variance of $st_0 st_0$. In the same manner, we can derive for $st_1 st_1$ the expression

$$
E(x_i' - \mu)(x_j' - \mu)
$$

$$
= \frac{1}{k_1 k_2 (n_1 n_2 - 1)} \left[\frac{1}{k_1 k_2 n_1 n_2} \sum \sum (k_1 n_1 - |u|)(k_2 n_2 - |v|)\rho_{uv} \right.
$$

$$
- \frac{1}{k_1 k_2 n_1} \sum \sum (k_1 n_1 - |u|)(k_2 - |v|)\rho_{uv}
$$

(15)
$$
- \frac{1}{k_1 k_2 n_2} \sum \sum (k_1 - |u|)(k_2 n_2 - |v|)\rho_{uv} + \frac{1}{k_1 k_2} \sum \sum (k_1 - |u|)
$$

$$
\cdot (k_2 - |v|)\rho_{uv} + \frac{2(k_1 k_2 n_1 - 1)}{k_1 n_1 (k_1 n_1 - 1)} \sum_{u=1}^{k_1 n_1} (k_1 n_1 - u)\rho_{0v}
$$

$$
- \frac{2(k_1 k_2 - 1)}{k_1 (k_1 - 1)} \sum_{u=1}^{k_1} (k_1 - u)\rho_{uv} + \frac{2(k_1 k_2 n_2 - 1)}{k_2 n_2 (k_2 n_2 - 1)} \sum_{v=1}^{k_2 n_2} (k_2 n_2 - v)\rho_{0v}
$$

$$
\left. - \frac{2(k_1 k_2 - 1)}{k_2 (k_2 - 1)} \sum_{v=1}^{k_2} (k_2 - v)\rho_{uv} \right].
$$

Thus we can evaluate $\sigma^2(X)$ for all types of stratified random sampling.

5c. *Systematic sampling*. In a similar manner to that used for stratified random sampling, we can use (13) to evaluate the variances of systematic sampling. Values of $E(x_i' - \mu)(x_j' - \mu)$ for three of the possible methods of sampling are given below. For $sy_1 sy_1$

$$
(16) \quad E(x_i' - \mu)(x_j' - \mu) = \frac{1}{n_1 n_2 (n_1 n_2 - 1)} \sum \sum (n_1 - |u|)(n_2 - |v|)\rho_{k_1 u, k_2 v}
$$

For $sy_1 r_0$

$$
E(x_i' - \mu)(x_j' - \mu) = \frac{1}{k_2^2 n_1 n_2 (n_1 n_2 - 1)} \sum \sum (n_1 - |u|)
$$

(17)
$$
\cdot (k_2 n_2 - |v|)\rho_{k_1 u, v} - \frac{2(k_2 - 1)}{k_2^2 n_2 (n_1 n_2 - 1)(k_2 n_2 - 1)} \sum_{v=1}^{k_2 n_2} (k_2 n_2 - v)\rho_{0v}.
$$

For $sy_0 sy_0$

$$
E(x_i' - \mu)(x_j' - \mu) = \frac{1}{k_1 k_2 (n_1 n_2 - 1)} \left[\frac{1}{k_1 k_2 n_1 n_2} \sum \sum (k_1 n_1 - |u|) \right.
$$

$$
\cdot (k_2 n_2 - |v|)\rho_{uv} - \frac{1}{k_1 k_2 n_1} \sum \sum (k_1 n_1 - |u|)(k_2 - |v|)\rho_{uv}
$$

$$
- \frac{1}{k_1 k_2 n_2} \sum \sum (k_1 - |u|)(k_2 n_2 - |v|)\rho_{uv} + \frac{1}{k_1 k_2} \sum \sum (k_1 - |u|)
$$

(18)
$$
\left. \cdot (k_2 - |v|)\rho_{uv} + \frac{k_1}{k_2 n_1} \sum \sum (n_1 - |u|)(k_2 - |v|)\rho_{k_1 u, v} \right.
$$

$$- \frac{2k_1}{k_2} \sum_{v=1}^{k_2} (k_2 - v)\rho_{0v} + \frac{k_2}{k_1 n_2} \sum \sum (k_1 - |u|)(n_2 - |v|)\rho_{u_1 k_2 v}$$

$$- \frac{2k_2}{k_1} \sum_{u=1}^{k_1} (k_1 - u)\rho_{u0} \Bigg].$$

The derivation of (18) may be compared with that of (15).

6. Effect of alignment. We can examine the effect of alignment either by an examination of the values of the variance of different samples, or by the direct use of (13). For random and stratified random sampling, the effect of alignment is to increase the variance of the sample by an amount

$$\Sigma\Sigma \, a_{uv}(\rho_{0v} - \rho_{uv}) + \Sigma\Sigma \, b_{uv}(\rho_{u0} - \rho_{uv}) \qquad \text{where } a_{uv} \geq 0,$$
$$b_{uv} \geq 0.$$

This will be positive for monotonic decreasing correlation functions, and for the majority of functions realised in practice. Thus alignment will usually increase the variance for random and stratified random samples.

For systematic samples, the position is more complicated, but, roughly, the variance is increased by an amount

$$\Sigma\Sigma \, a_{uv}(\rho_{k_1 u, k_2 v} - \bar{\rho}_{k_1 u, k_2 v}),$$

where $a_{uv} \geq 0$ and $\bar{\rho}_{k_1 u, k_2 v}$ is a mean over a rectangle, centre $\rho_{k_1 u, k_2 v}$ for u and v non-zero, and is a mean over a line, length k_1 centre $\rho_{0, k_2 v}$ for u zero, (and similarly for v zero). Whether this is positive or negative will depend on the correlation function, and it will have to be investigated for the types of correlation function which are encountered.

7. Limiting forms. For a continuous process, when n_1 and n_2 are large, we may, in the same manner as for linear sampling, obtain integral approximations to the sampling variance, provided that $\Sigma\Sigma \, \rho_{d_1 u, d_2 v}$ converges.

We thus have

$$(19) \qquad \sigma^2(r_0 \, r_0) = \sigma^2(st_0 \, r_0) \sim \sigma^2/n_1 n_2,$$

$$(20) \qquad \sigma^2(r_1 \, r_0) \sim \frac{\sigma^2}{n_1 n_2} \left[1 + \frac{2}{d_2} \int_0^\infty \rho_{0v} \, \delta v \right].$$

$$(21) \qquad \sigma^2(r_1 \, r_1) \sim \frac{\sigma^2}{n_1 n_2} \left[1 + \frac{2}{d_2} \int_0^\infty \rho_{0v} \, \delta v + \frac{2}{d_1} \int_0^\infty \rho_{u0} \, \delta u \right],$$

$$(22) \qquad \sigma^2(st_1 \, r_0) \sim \frac{\sigma^2}{n_1 n_2} \left[1 - \frac{1}{d_1^2 d_2} \int_{-\infty}^\infty \int_{-d_1}^{d_1} (d_1 - |u|)\rho_{uv} \, \delta u + \frac{2}{d_2} \int_0^\infty \rho_{0v} \, \delta v \right],$$

$$(23) \qquad \sigma^2(st_0 \, st_0) \sim \frac{\sigma^2}{n_1 n_2} \left[1 - \frac{1}{d_1^2 d_2^2} \int_{-d_2}^{d_2} \int_{-d_1}^{d_1} (d_1 - |u|)(d_2 - |v|)\rho_{uv} \, \delta u \, \delta v \right],$$

$$\sigma^2(st_1\, st_1) \sim \frac{\sigma^2}{n_1\, n_2}\left[1 - \frac{1}{d_1^2\, d_2}\int_{-\infty}^{\infty}\int_{-d_1}^{d_1}(d_1 - |\,u\,|)\rho_{uv}\,\delta u\,\delta v\right.$$

$$- \frac{1}{d_1\, d_2^2}\int_{-d_2}^{d_2}\int_{-\infty}^{\infty}(d_2 - |\,v\,|)\rho_{uv}\,\delta u\,\delta v + \frac{1}{d_1^2\, d_2^2}\int_{-d_2}^{d_2}\int_{-d_1}^{d_1}(d_1 - |\,u\,|)$$

(24)

$$\cdot (d_2 - |\,v\,|)\rho_{uv}\,\delta u\,\delta v + \frac{2}{d_1}\int_0^{\infty}\rho_{u0}\,\delta u - \frac{2}{d_1^2}\int_0^{d_1}(d_1 - u)\rho_{u0}\,\delta u$$

$$\left. + \frac{2}{d_2}\int_0^{\infty}\rho_{0v}\,\delta v - \frac{2}{d_2^2}\int_0^{d_2}(d_2 - v)\rho_{0v}\,\delta v\right],$$

$$\sigma^2(sy_1\, r_0) \sim \frac{\sigma^2}{n_1\, n_2}\left[1 - \frac{1}{d_1\, d_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_{uv}\,\delta u\,\delta v\right.$$

(25)

$$\left. + \frac{1}{d_2}\sum_{u=-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_{d_1 u, v}\,\delta v - \frac{2}{d_2}\int_0^{\infty}\rho_{0v}\,\delta v\right],$$

(26) $\quad \sigma^2(sy_1\, sy_1) \sim \dfrac{\sigma^2}{n_1\, n_2}\left[\displaystyle\sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty}\rho_{d_1 u, d_2 v} - \dfrac{1}{d_1\, d_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho_{uv}\,\delta u\,\delta v\right],$

$$\sigma^2(sy_0\, sy_0) \sim \frac{\sigma^2}{n_1\, n_2}\left[1 - \frac{1}{d_1^2\, d_2}\int_{-\infty}^{\infty}\int_{-d_1}^{d_1}(d_1 - |\,u\,|)\rho_{uv}\,\delta u\,\delta v\right.$$

$$- \frac{1}{d_1\, d_2^2}\int_{-d_2}^{d_2}\int_{-\infty}^{\infty}(d_2 - |\,v\,|)\rho_{uv}\,\delta u\,\delta v$$

(27)

$$+ \frac{1}{d_1^2\, d_2^2}\int_{-d_2}^{d_2}\int_{-d_1}^{d_1}(d_1 - |\,u\,|)(d_2 - |\,v\,|)\rho_{uv}\,\delta u\,\delta v$$

$$+ \frac{1}{d_2^2}\sum_{u=-\infty}^{\infty}\int_{-d_2}^{d_2}(d_2 - |\,v\,|)\rho_{d_1 u, v}\,\delta v - \frac{1}{d_2^2}\int_{-d_2}^{d_2}(d_2 - |\,v\,|)\rho_{0v}\,\delta v$$

$$\left. + \frac{1}{d_1^2}\sum_{v=-\infty}^{\infty}\int_{-d_1}^{d_1}(d_1 - |\,u\,|)\rho_{u, d_2 v}\,\delta u - \frac{1}{d_1^2}\int_{-d_1}^{d_1}(d_1 - |\,u\,|)\rho_{u0}\,\delta u\right].$$

8. Particular case where $\rho_{uv} = \rho_u \rho_v$. We note that, if $\rho_{uv} = \rho_u \rho_v$[5] most of these forms can be simplified greatly. If we write

$$sy_u = 1 - \frac{2}{d_1}\int_0^{\infty}\rho_u\,\delta u + 2\sum_{u=1}^{\infty}\rho_{d_1 u},$$

$$st_u = 1 - \frac{2}{d_1^2}\int_0^{d_1}(d_1 - u)\rho_u\,\delta u,$$

with similar forms for sy_v and st_v, and, also

$$f_1 = \frac{2}{d_2}\int_0^{\infty}\rho_v\,\delta v, \qquad f_1' = \frac{2}{d_2^2}\int_0^{d_2}(d_2 - v)\rho_v\,\delta v, \qquad f_1'' = 2\sum_{v=1}^{\infty}\rho_{d_2 v},$$

$$f_2 = \frac{2}{d_1}\int_0^{\infty}\rho_u\,\delta u, \qquad f_2' = \frac{2}{d_1^2}\int_0^{d_1}(d_1 - u)\rho_u\,\delta u, \qquad f_2'' = 2\sum_{u=1}^{\infty}\rho_{d_1 u},$$

---

[5] A sufficient condition for this to be a valid autocorrelation function is that both ρ_u and ρ_v should be autocorrelation functions.

then we have, for example,

$$(28) \qquad \sigma^2(r_1 r_0) \sim \frac{\sigma^2}{n_1 n_2} (1 + f_1),$$

$$(29) \qquad \sigma^2(r_1 r_1) \sim \frac{\sigma^2}{n_1 n_2} (1 + f_1 + f_2),$$

$$(30) \qquad \sigma^2(st_0 st_0) \sim \frac{\sigma^2}{n_1 n_2} (st_u st_v + st_u + st_v),$$

$$(31) \qquad \sigma^2(st_1 st_1) \sim \frac{\sigma^2}{n_1 n_2} (st_u st_v + f_1 st_u + f_2 st_v),$$

$$(32) \qquad \sigma^2(sy_1 sy_1) \sim \frac{\sigma^2}{n_1 n_2} (sy_u sy_v + f_1 sy_u + f_2 sy_v),$$

$$(33) \qquad \sigma^2(sy_0 sy_0) \sim \frac{\sigma^2}{n_1 n_2} (st_u st_v + f_1' sy_u + f_2' sy_v).$$

From these we get

$$(34) \qquad \sigma^2(st_1 st_1) - \sigma^2(sy_1 sy_1) \sim \frac{\sigma^2}{n_1 n_2}$$
$$[(st_u st_v - sy_u sy_v) + f_1(st_u - sy_u) + f_2(st_v - sy_v)],$$

$$(35) \qquad \sigma^2(sy_1 sy_1) - \sigma^2(st_0 st_0) \sim \frac{\sigma^2}{n_1 n_2}$$
$$[\{(1 - sy_u)(1 - sy_v) - (1 - st_u)(1 - st_v)\} + f_1'' sy_u + f_2'' sy_v],$$

$$(36) \qquad \sigma^2(st_0 st_0) - \sigma^2(sy_0 sy_0) \sim \frac{\sigma^2}{n_1 n_2} [f_1'(st_u - sy_u) + f_2'(st_v - sy_v)].$$

The forms (34), (35) and (36) enable us to compare the variances of the samples in two dimensions by using the one-dimensional results. For most practical cases, we know that the f's are positive, $st_u \geq sy_u$ and $st_v \geq sy_v$, so that

$$(37) \qquad \sigma^2(st_1 st_1) \geq \sigma^2(sy_1 sy_1) \geq \sigma^2(st_0 st_0) \geq \sigma^2(sy_0 sy_0).$$

The values of $\sigma^2(st_0 st_0)/\sigma^2(r_0 r_0)$, $\sigma^2(sy_1 sy_1)/\sigma^2(r_0 r_0)$, $\sigma^2(sy_0 sy_0)/\sigma^2(r_0 r_0)$ and $\sigma^2(st_0 st_0)/\sigma^2(sy_0 sy_0)$ for $\rho_{d_1 u} = \rho_1^{|u|}$ and $\rho_{d_2 v} = \rho_2^{|v|}$ are given in table 3. It is not difficult to show that for a given number of samples, (d_1, d_2 fixed), $\sigma^2(st_0 st_0)$, $\sigma^2(sy_1 sy_1)$ and $\sigma^2(sy_0 sy_0)$ are least when $\rho_1 = \rho_2$. The expressions tabulated have a value of 1 for $\rho_1 = \rho_2 = 0$ and tend to limiting values of 0, 2/3, 0, and 2 respectively as ρ_1 and ρ_2 tend to 1. It is interesting to note that for ρ_1 and ρ_2 differing by more than 0.4 the grid imposed by $sy_1 sy_1$ is less efficient than purely random sampling. The type of function $\rho_{uv} = \rho_u \rho_v$[6] is, however, less likely to be realised

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[6] For a town survey, we might find the correlation between two points depending on a within-streets and a between-streets correlation, so that this function could be realised.

TABLE 3

Comparison of the efficiencies of systematic and random sampling for various values of ρ_1 and ρ_2

ρ_2 \ ρ_1	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.000	1.222	1.500	1.857	2.333	3.000	4.000	5.667	9.000	19.000	∞
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.1		0.720	0.669	0.632	0.601	0.575	0.551	0.529	0.508	0.489	0.471
		0.739	0.754	0.827	0.956	1.160	1.488	2.055	3.215	6.734	∞
		0.596	0.534	0.493	0.462	0.437	0.416	0.398	0.382	0.367	0.354
		1.21	1.25	1.28	1.30	1.31	1.32	1.33	1.33	1.33	1.33
0.2			0.609	0.565	0.529	0.497	0.469	0.443	0.419	0.396	0.375
			0.706	0.721	0.788	0.914	1.134	1.532	2.362	4.911	∞
			0.462	0.416	0.380	0.352	0.328	0.307	0.289	0.272	0.257
			1.32	1.36	1.39	1.41	1.43	1.44	1.45	1.46	1.46
0.3				0.516	0.476	0.441	0.409	0.380	0.354	0.328	0.305
				0.689	0.707	0.778	0.924	1.209	1.825	3.751	∞
				0.365	0.327	0.297	0.271	0.249	0.229	0.212	0.196
				1.41	1.45	1.49	1.51	1.53	1.54	1.55	1.55
0.4					0.432	0.394	0.360	0.329	0.300	0.272	0.247
					0.680	0.702	0.787	0.983	1.437	2.900	∞
					0.288	0.256	0.229	0.206	0.185	0.167	0.151
					1.50	1.54	1.57	1.60	1.62	1.63	1.64
0.5						0.354	0.317	0.284	0.253	0.223	0.196
						0.675	0.703	0.821	1.139	2.228	∞
						0.223	0.195	0.171	0.150	0.132	0.115
						1.59	1.63	1.66	1.68	1.70	1.71
0.6							0.279	0.243	0.210	0.180	0.151
							0.671	0.712	0.908	1.679	∞
							0.167	0.142	0.121	0.102	0.085
							1.67	1.71	1.74	1.76	1.78
0.7								0.206	0.172	0.139	0.109
								0.669	0.742	1.226	∞
								0.118	0.096	0.077	0.059
								1.75	1.79	1.82	1.84
0.8									0.136	0.102	0.070
									0.667	0.863	∞
									0.074	0.055	0.037
									1.84	1.87	1.89
0.9										0.067	0.034
										0.667	∞
										0.035	0.018
										1.92	1.95

in practice than a centrally-symmetric function, which is independent of the choice of axes. For this reason, we consider next this latter type of function.

9. Centrally-symmetric correlation functions. Dedebant and Wehrte [3] have given a necessary and sufficient condition for $\rho(u, v)$ to be a correlation function as

$$(38) \qquad \rho(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos (\omega u - \mu_v) \delta F(\omega, \mu),$$

or alternatively,

$$(39) \qquad f(\omega, \mu) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos (\omega u - \mu_v) \rho(u, v) \, \delta u \, \delta v.$$

For a centrally-symmetric correlation function we can put $u = r \cos \theta, v = r \sin \theta$ then $\rho(u, v) = \rho(r)$ and

$$f(\omega, \mu) = \frac{1}{(2\pi)^2} \int_{0}^{\infty} \int_{0}^{2\pi} \cos (r \sqrt{\omega^2 + \mu^2} \cos \theta_1) \rho(r) r \, d\theta_1 \, dr,$$

$$\text{where} \quad \theta_1 = \theta + \tan^{-1}(\mu/\omega),$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} J_0(r\tau) \rho(r) r \, dr, \quad \text{where} \quad \tau = \sqrt{\omega^2 + \mu^2}.$$

Thus, if $\rho(u, v)$ is centrally-systematic, then so is $f(\omega, \mu)$ and conversely, so that we get

$$(40) \qquad f(\tau) = \frac{1}{2\pi} \int_{0}^{\infty} J_0(r\tau) \rho(r) r \delta r,$$

and

$$(41) \qquad \rho(r) = 2\pi \int_{0}^{\infty} J_0(r\tau) f(\tau) \tau \delta \tau.$$

We can thus find suitable forms for $\rho(r)$ and $f(\tau)$. In this connection the formula $\int_{0}^{\infty} J_0(yz) e^{-ay} \delta y = 1/(a^2 + z^2)^{1/2}, a \geq 0$, is useful, since we can see that $\frac{\delta^n}{\delta a^n}(e^{-ay}/y)$ and $\frac{\delta^n}{\delta a^n}(a^2 + z^2)^{-1/2}$ are possible functions for $2\pi f(\tau)$ and $\rho(r)$ although our choice must be limited by the stochastic nature of $\rho(r)$ as well as by its convergence. Thus, for example, $a = n = 0$ gives $1/2\pi\tau$ and $1/r$ as spectral and correlation functions, but these will not converge.

In the linear case, the Markoff process $\rho(u) = e^{-au}$ had a spectral function $f(\tau) = 1/\pi(a^2 + \tau^2)$ which is a Cauchy distribution in one dimension. If we take a two-dimensional Cauchy distribution[7] as our spectral function we get $f(\tau) =$

---

[7] In the same way as the ordinary Cauchy distribution can be considered as a density distribution on a line produced by a point source at a distance a, radiating in all directions, so can a two-dimensional distribution be considered as a density distribution on a plane from a source at distance a.

$a/2\pi(a^2 + \tau^2)^{3/2}$ and $\rho(r) = -\dfrac{\delta}{\delta a}(e^{-ar}/r) = e^{-ar}$. Thus it appears that a generalised Cauchy distribution will be the spectral function for a generalised Markoff process.

We can, of course, consider an "elliptical" Markoff process given by[8]

$$(42) \qquad \rho(u, v) = \exp -\left[\frac{u^2}{a^2} - \frac{2muv}{ab} + \frac{v^2}{b^2}\right]^{1/2}$$

but, in what follows, to simplify the computation, m will be taken as zero, so that by changing the units in which d_1 and d_2 are measured, we will work with a process $\rho(r) = e^{-ar}$.

TABLE 4

Comparison of observed serial correlations with theoretical values obtained from a centrally-symmetric correlation function

Distance in miles	Rows		Columns		North-east		South-east	
	Observed	Calculated	Observed	Calculated	Observed	Calculated	Observed	Calculated
1	0.332	0.368	0.310	0.368	—	—	—	—
2	—	—	—	—	0.264	0.243	0.264	0.243
2	0.149	0.135	0.090	0.135	—	—	—	—
$2\sqrt{2}$	—	—	—	—	0.050	0.059	0.129	0.059
3	0.009	0.050	−0.029	0.050	—	—	—	—
$3\sqrt{2}$	—	—	—	—	−0.050	0.018	0.070	0.018
4	0.034	0.018	−0.041	0.018	—	—	—	—
$4\sqrt{2}$	—	—	—	—	−0.020	0.004	0.060	0.004

This process does not seem to be far removed from the type of correlation function experienced in agricultural field work.[9] Osborne [4] has mentioned the possible use of $\rho_u = e^{-\lambda u}$. Mahalanobis [5] has calculated correlations for a paddy field of 800 cells; his values are shown in table 4, together with values of the function e^{-r}. Bearing in mind that the standard error of each of Mahalanobis' values is approximately 0.035, the fit is seen to be quite good, although an elliptical process with axes running south-east and north-east would undoubtedly fit the observations better.

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[8] In this light, $\rho(r) = e^{-ar}$ will be called the circular Markoff process, while $\rho_{uv} = \rho_1^{|u|}\rho_2^{|v|}$ and $\rho_{uv} = \exp\left\{-\left|\dfrac{u}{a} + \dfrac{v}{b}\right|\right\}$ will be known as degenerate Markoff processes of the first and second orders.

[9] This is further supported by the fact that using a function of this kind it is possible to obtain numerically a law in substantial agreement with Fairfield-Smith's law over a wide range of values.

10. The relative efficiencies of systematic and stratified random sampling.
Ideally the correlation functions developed in the last section should be used
in the expression (19)–(27), but these functions are not capable of easy integra-
tion. An alternative approach can be made if we note that

$$
\frac{\sigma^2(st_0\,st_0) - \sigma^2(sy_0\,sy_0)}{\sigma^2(r_0\,r_0)} \sim \frac{1}{d_1} \int_{-d_1}^{d_1} \left(1 - \frac{|u|}{d_1}\right) F(u, d_2)\delta u
$$

(43)

$$
+ \frac{1}{d_2} \int_{-d_2}^{d_2} \left(1 - \frac{|v|}{d_2}\right) F(v, d_1)\delta v
$$

where

$$
F(u, d_2) = \frac{2}{d_2} \left[\int_0^{d_2} \frac{v}{d_2}\, \rho_{uv}\, \delta v + \int_{d_2}^{\infty} \rho_{uv}\, \delta v - d_2 \sum_{v=1}^{\infty} \rho_{u,d_2v} \right],
$$

$$
F(v, d_1) = \frac{2}{d_1} \left[\int_0^{d_1} \frac{u}{d_1}\, \rho_{uv}\, \delta u + \int_{d_1}^{\infty} \rho_{uv}\, \delta u - d_1 \sum_{u=1}^{\infty} \rho_{d_1u,v} \right].
$$

It is seen that $F(u, d_2)$ and $F(v, d_1)$ are extensions of the expressions obtained for
$(\sigma^2_{st} - \sigma^2_{sy})/\sigma^2_r$ in section 2. Hence, if $F(u, d_2)$ and $F(v, d_1)$ are both positive
functions, systematic sampling is more accurate than stratified random sampling.
A particular case of this occurs when $\rho_{uv} = \rho_1^u \rho_2^v$. However when $\rho_{uv} = \exp$
$\{-(u^2 + v^2)^{1/2}\}$, $F(u, d_2)$ is not always positive, since, as u increases, ρ_{uv} becomes a
convex function of v. This complicates the interpretation of (43) greatly since it
appears that as u varies from 0 to d_1, $F(u, d_2)$ varies from $+ \infty$ to an unknown
value X. This value will be positive if $d_2 \gg d_1$ and negative if $d_1 \gg d_2$ so
that if the sampling is disproportionate in the two directions systematic sampling
will be more efficient than stratified random sampling. Furthermore, if $d_1 = d_2 = d$
and $d \to 0$, $F(u, d) \to \infty$ and systematic sampling again appears to be more
efficient. Thus in a wide variety of cases this type of systematic sampling i.e.
sy_0sy_0 gives a more accurate result than random sampling.

11. Estimation of sampling errors. An examination of formulas (7)–(18)
shows that the principles used for the estimation of linear errors can be used in
plane sampling. If we consider that each sample can be broken up into inde-
pendent units each of which is situated in one of s strata, then for q replications
we will have $qr - s$ degrees of freedom for error. For example, r_0r_0, r_0r_1, st_0r_0
and st_0r_1 will have $qn_1n_2 - 1$, $qn_2 - 1$, $qn_1n_2 - n_1$ and $qn_2 - 1$ degrees of freedom
respectively, so that a single sample will contain an unbiased estimate of error,
but st_0st_0, st_0st_1, st_1st_1, sy_0sy_0 and sy_1sy_1 will have $n_1n_2(q - 1)$, $n_2(q - 1)$, $q - 1$
and $q - 1$ degrees of freedom and will require replication to form a valid estimate
of error. We can however use the method of splitting our sample into several
parts each of which will give a fairly accurate estimate of error. We may, again,
consider the possibility of using a set of systematic samples, which are evenly
spaced, to estimate the sampling error, and we will see that the exclusion of the
ρ's of lower order may lead to appreciable bias unless the correlation between

successive terms of the sample is small, but, as Yates has pointed out, this method will provide an upper limit for our sampling error. These methods of sampling are illustrated by the examples given below.

12. Examples. We shall consider the three methods of estimating the sampling errors of a systematic sample:

(1) using sets of systematic samples randomly placed with respect to each other, i.e. the material to be sampled is broken up into a series of sub-areas or blocks and several systematic samples are taken in each block; the error variance is calculated from the variances of the systematic samples in each block,

(2) using one set of systematic samples randomly placed, i.e. several systematic samples are taken and the area is then broken up into sub-areas or blocks; the error variance is calculated from the variances of the portions of the systematic samples in each block,

(3) using one systematic sample i.e. one systematic sample is taken which is broken into several systematic samples of wider spacing, e.g. four samples at four times the original spacing, the area is then divided into several sub-areas and the error variance is calculated from the variances of the portions of the sub-systematic samples in each block.

These three methods are increasingly accurate in their estimation of the mean, increasingly biased in their estimation of the sampling variance, and decreasingly difficult in their practical application, so that our method of sampling may vary according to the population and according to the use to which the results are to be put. It is, for example, conceivable that subsequent sampling will yield an improved estimate of error so that initially only a rough guide may be required.

a. If we are sampling from a continuous linear population with a large number of observations in each part into which we split our series, methods (1) and (2) will both give accurate estimates of the variance per term

$$\sigma^2 \left(1 - \frac{2}{d} \int_0^\infty \rho_u \, \delta u + 2 \sum_{u=1}^\infty \rho_{du} \right).$$

Method (3) will, however, estimate σ^2 instead of the correct variance per term, which is

$$\sigma^2 \left(1 - \frac{2q}{d} \int_0^\infty \rho_u \, \delta u + 2 \sum_{u=1}^\infty \rho_{du/q} \right).$$

Thus the estimates of sampling variance by method (3) will in general be higher than the estimates by methods (1) and (2), although the actual variance will be lower.

b. Kendall [6, 7] has constructed 480 terms of an artificial series $u_{n+2} = 1.1 \, u_{n+1} - 0.5 \, u_n + \epsilon_{n+2}$ where the ϵ_n are rectangularly distributed from -49 to 49. For this series $\sigma^2 = 2379.81$ and $s^2 = 2535.11$. The series was split in six parts of 80 terms, for each of which $n = 5$, $k = 16$, $q = 4$, so that 18 degrees of freedom were available for error. The results for this sampling configuration are

given in table 5. The values in this table corroborate the conclusions for large samples of continuous populations.

c. A number of uniformity trials were taken and sampled according to the systems st_1st_1 and sy_1sy_1. For sampling according to the system st_1st_1 the error

TABLE 5
Comparison of three methods of estimating the sampling error of systematic samples for an autoregressive scheme

Method	Estimate of sampling variance per term, s^2, based on 18 degrees of freedom	$E\,(s^2)$	True sampling variance per term
(1)	3228	2170	2170
(2)	1872	2170	2167
(3)	3709	2577	423

TABLE 6
Comparison of efficiencies of different methods of sampling on three uniformity trials

Source	Kalamkar [8]			Wiebe [9]			Wynne Sayes and Karishna, Iyez [10]		
No. In Cochran's [11] Catalogue	72			132			108		
Crop	Potatoes			Wheat			Sugar cane		
No. of Plots	576			1440			960		
Mean	23.262			587.95			270.89		
Variance per term	15.555			10,918.0*			1794.42		
Type of sampling	$st_1\,st_1$	$sy_1\,sy_1$	$sy_1\,sy_1$	$st_1\,st_1$	$sy_1\,sy_1$	$sy_1\,sy_1$	$st_1\,st_1$	$sy_1\,sy_1$	$sy_1\,sy_1$
Proportion sampled	1/6	1/6	1/6	1/9	1/9	1/9	1/8	1/8	1/8
Method of estimating error		(2)	(3)		(2)	(3)		(2)	(3)
No. of partitions	1	4	4	1	4	4	1	5	5
n_1	3	3	6	4	2	4	4	2	4
k_1	2	2	1	3	6	3	2	4	2
n_2	16	2	4	20	5	10	15	3	6
k_2	6	12	6	6	6	3	8	8	4
q	2	4	1	2	4	1	2	4	1
Mean	23.140	23.435	23.323	586.54	598.65	275.29	275.29	266.72	271.27
Estimated variance per term	9.763	2.689	4.889	5151.6	5772.7	7038.5	1320.15	799.29	1269.54
Degrees of freedom of estimated variance	48	12	12	80	12	12	60	15	15

* Based on the original 1500 plots.

was estimated by taking two samples per strata, while, for sampling according to the system sy_1sy_1, the error was estimated by comparing sets of four samples in each part of the series by methods (2) and (3). The results of this sampling are shown in table 6. While the number of trials is small, the trend to be seen in the results agrees very well with the conclusions reached above.

13. Trend in the population. Frequently in taking samples from a population, we are faced with the problem of a trend. This will not greatly affect random and stratified random samples as estimates of the population mean, but the efficiency of systematic samples will be affected to a large extent. If we consider linear sampling, and denote by S_i the sample whose first element is x_i then the set of samples S_i will usually be monotonic with i and the difference between S_1 and S_k will be large (roughly equal to $x_1 - x_k$).

Yates [1] has suggested a method to overcome this difficulty; by letting S_i represent

$$\frac{1}{n-1}\left[\frac{i}{k}x_i + x_{i+k} + \cdots + x_{i+(n-2)k} + \frac{k-i}{k}x_{n+(n-1)k}\right],$$

the difference between systematic samples due to trend is largely removed. It is easily seen that this necessitates a small loss of information, and in particular, for a continuous random population the variance is $(n - \frac{2}{3})\sigma^2/(n-1)^2$ instead of σ^2/n. For plane samples, the corresponding adjusted sample will be

$$S_{ij} = \frac{1}{(n_1-1)(n_2-1)}\left[\frac{ij}{k_1 k_2}x_{ij} + \frac{j}{k_2}x_{i+k_1,j} + \cdots + \frac{j(k_1-i)}{k_1 k_2}x_{i+(n_1-1)k_1,j}\right.$$

$$+ \frac{i}{k_1}x_{i,j+k_2} + x_{i+k_1,j+k_2} + \cdots + \frac{(k_1-i)}{k_1}x_{i+(n_1-1)k_1,j+k_2}$$

$$\left.+ \frac{i(k_2-j)}{k_1 k_2}x_{i,j+(n_2-1)k_2} + \cdots + \frac{(k_1-i)(k_2-j)}{k_1 k_2}x_{i+(n_1-1)k_1,j+(n_2-1)k_2}\right]$$

with a similar loss of information.

Trend is, however, most likely to be appreciable in large samples, and in this case, the loss of information due to end adjustments is negligible, so that the conclusions reached above will remain unaltered.

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