A NOTE ON FISHER'S INEQUALITY FOR BALANCED INCOMPLETE BLOCK DESIGNS

By R. C. Bose

Institute of Statistics, University of North Carolina

1. An experimental design in which v varieties or treatments are arranged in b blocks, is called a *balanced incomplete block design* if

(i) Each block has exactly k treatments ($k < v$) no treatment occurring twice in the same block.

(ii) Each treatment occurs in exactly r blocks.

(iii) Any two treatments occur together in exactly λ blocks.

It is easy to see that the parameters v, b, r, k, λ of the design satisfy the relations

$$bk = vr \tag{1.0}$$

$$\lambda(v - 1) = r(k - 1). \tag{1.1}$$

Also it is readily seen that

$$r > \lambda \tag{1.2}$$

for otherwise with any given treatment every other treatment would occur in every block. This would make $k = v$, and the design would become a 'randomised block design'.

Fisher (1940), showed that a necessary condition for the existence of a balanced incomplete block design with v treatments and b blocks is

$$b \geqq v. \tag{1.3}$$

It is the object of this note to give a very simple proof of Fisher's inequality.

2. Consider a balanced incomplete block design with parameters

$$v, b, r, k, \lambda \tag{2.0}$$

and let

$$n_{ij} = 1 \text{ or } 0 \tag{2.1}$$

according as the ith treatment does or does not occur in the jth block. Clearly

$$\sum_{j=1}^{b} n_{ij}^2 = r \tag{2.2}$$

$$\sum_{j=1}^{b} n_{ij}n_{i'j} = \lambda \qquad (i \neq i'). \tag{2.3}$$

If possible let $b < v$. Consider the $v \times v$ matrix

$$N = \begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1b} & 0 & \cdots & 0 \\ n_{21} & n_{22} & \cdots & n_{2b} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ n_{v1} & n_{v2} & \cdots & n_{vb} & 0 & \cdots & 0 \end{bmatrix} \tag{2.4}$$

where the last $v - b$ columns of N consist of zeros. It follows from (2.2) and (2.3) that

$$NN' = \begin{bmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots \\ \lambda & \lambda & \cdots & r \end{bmatrix} \tag{2.5}$$

where N' denotes the transpose of N.

$$\det (NN') = \{r + \lambda(v - 1)\} (r - \lambda)^{v-1} \tag{2.6}$$

But

$$= kr(r - \lambda)^{v-1} \text{ from (1.1).}$$

$$\det (NN') = \det N \det N' = 0. \tag{2.7}$$

This makes $r = \lambda$, and contradicts (1.2). Hence the assumption $b < v$ is wrong, and we must have

$$b \geq v \tag{2.8}$$

REFERENCES

[1] R. A. Fisher, "An examination of the different possible solutions of a problem in incomplete blocks," *Annals of Eugenics*, London, Vol. 10 (1940), pp. 52–75.

[2] F. Yates, "Incomplete randomised blocks," *Annals of Eugenics*, London, Vol. 7 (1936), pp. 121–140.

ABSTRACTS OF PAPERS

(Presented September 1, 1949 at Boulder at the Twelfth Summer Meeting of the Institute)

1. Structure of Statistical Elements. Duane M. Studley, Foundation Research, Colorado Springs, Colorado.

Research in logical semantics and in practical elementation has set forth the proposition that all words and ideas have set form. As a consequence of this universal proposition all notions and conceptions in statistics should be accessible to set-theoretic analysis and interpretation. This paper explains the results of a preliminary analysis performed on statistical notions and conceptions with a view to a proper organization of definitions and conceptions which will, it is hoped, make possible a better and simpler construction of statistics from a system of basic notions.