ON CRAIG’S THEOREM

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Since \( \sigma^2 > \nu^2 \), this is the characteristic function for two variables which are normally distributed. Thus, the simultaneous distribution of \( \xi \) and \( M \) is asymptotically normal. It is of interest to note that, if the pdf \( f(x) \) is symmetric, the correlation coefficient is zero, and \( M \) and \( \xi \) are asymptotically independent. We might also note that \( \phi(t_1, 0) \) is the characteristic function for the mean deviation from the sample median. Thus, the random variable \( M \) is asymptotically normal with asymptotic mean and variance \( \nu' \) and \( \left( (m - \theta)^2 + \sigma^2 - \nu^2 \right)/2k \) respectively.

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NOTE ON THE EXTENSION OF CRAIG’S THEOREM TO NON-CENTRAL VARIATES

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The following notation is used: \( A, A_1, A_2 \) are real symmetric matrices, \( L \) is an orthogonal matrix, \( \Gamma \) is a diagonal matrix of latent roots, and \( X, Y, M \) and \( U \) are column vectors.

**Theorem.** Let \( X' = (x_1, \cdots, x_n) \) be a set of normally and independently distributed variates with equal variance \( \sigma^2 \) and means \( M' = (m_1, \cdots, m_n) \). Then, a necessary and sufficient condition that a real symmetric quadratic form \( Q(X) = X'AX \) of rank \( r \) be distributed as \( \chi^2 \), where

\[
p(x^2, r, \lambda^2) = \frac{1}{2} e^{-x^2}(x^2/2)^{(r-2)/2} \frac{e^{-x^2/2}}{\Gamma[(r - 2)/2]},
\]

(1)

\[
\sum_{i=0}^{\infty} \frac{(\lambda^2 x^2/2)^i}{j! \Gamma[(r - 2j)/2]},
\]

is that \( A^2 = A \). If \( Q(X)/\sigma^2 \) is distributed by \( p(x^2, r, \lambda^2) \), then \( \lambda^2 = Q(M)/2\sigma^2 \).

Further, let \( Q_1(X) = X'A_1X \) and \( Q_2(X) = X'A_2X \) be real symmetric quadratic forms of ranks \( r_1 \) and \( r_2 \). Then a necessary and sufficient condition that \( Q_1(X) \) and \( Q_2(X) \) be statistically independent is that \( A_1A_2 = 0 \).

**Proof.** The theorem is proved by establishing the equivalence and factorization of moment generating functions [4]. The moment generating function of
\( p(x^2, r, \lambda^2) \) is
\[
G(t) = Ee^{t x^2/2} = e^{\lambda^2(1-t)(1-t)^{-\gamma/2}}.
\]

Let \( x_1, \ldots, x_n \) be normally and independently distributed with means \( E(x_i) = m_i \) and common variance \( \sigma^2 \). Without loss of generality, we may take \( \sigma^2 = 1 \), changing to the general case when necessary with the transformation \( x_i = z_i/\sigma \).

Let \( Q(X) = X'AX \) be a real symmetric quadratic form of rank \( r \). Then the moment generating function of \( Q(X) \) is
\[
G_0(t) = Ee^{tQ(X)/2} = (2\pi)^{-n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}(X-M)'(X-M) - X'TAX} \prod_1^n dx_i.
\]
If \( t \) is restricted to values such that \( |t| < |1/\gamma_0| \), where \( \gamma_0 \) is the dominant latent root of \( A \), then \( I - tA \) is positive definite and
\[
G_0(t) = (2\pi)^{-n/2} e^{\frac{1}{2}M'tA(I-tA)^{-1}M}
\]
\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}[X-(I-tA)^{-1}M]'(I-tA)[X-(I-tA)^{-1}M]} \prod_1^n dx_i = e^{\frac{1}{2}M'tA(I-tA)^{-1}M} |I - tA|^{-1}.
\]
If \( L \) is an orthogonal matrix such that
\[
L'AL = \Gamma = \begin{pmatrix}
\gamma_1 & 0 & \cdots & 0 \\
0 & \gamma_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \gamma_n
\end{pmatrix},
\]
where the \( \gamma_i \) are the latent roots of \( A \), then the transformation \( M = LU \) gives
\[
G_0(t) = e^{\frac{1}{2}U't\Gamma(I-t\Gamma)^{-1}U} |I - t\Gamma|^{-1}.
\]
A necessary and sufficient condition that \( G_0(t) = G(t) \) is that \( A^2 = A \). If \( A^2 = A \), then all of the latent roots of \( A \) are \( +1 \) or \( 0 \), and sufficiency can be established by substituting the appropriate value of each \( \gamma_i \) into equation (5), giving
\[
G_0(t) = e^{\frac{1}{2}U't\Gamma(I-t\Gamma)^{-1}U} |I - t\Gamma|^{-1} = G(t).
\]
Also \( \lambda^2 = \sum_i \gamma_i u_i^2/2 = \frac{1}{2}(U'\Gamma U) = \frac{1}{2}(M'AM) = Q(M)/2 \).

It is apparent from the form of \( G_0(t) \) that a necessary condition for \( G_0(t) = G(t) \) is that \( |I - tA|^{-1} = (1 - t)^{-\gamma/2} \). But it has been proved by Craig [1] that the condition \( A^2 = A \) is necessary, as well as sufficient, for this equality.

Next, let \( Q_1(X) = X'A_1X \) and \( Q_2(X) = X'A_2X \) be real symmetric quadratic forms of ranks \( r_1 \) and \( r_2 \). Then from (4)
\[
G(t_1, t_2) = Ee^{t_1Q_1/2 + t_2Q_2/2}
\]
\[
= e^{\frac{1}{2}M'(t_1A_1 + t_2A_2)'(I-t_1A_1 - t_2A_2)^{-1}M} |I - t_1A_1 - t_2A_2|^{-1},
\]
\( t_1, t_2 \) being restricted to values for which \((I - t_1 A_1 - t_2 A_2)\) is positive definite.

A necessary and sufficient condition that \(G(t_1, t_2) = G_Q(t_1) \cdot G_Q(t_2)\) is \(A_1 A_2 = 0\).

The required equation in the moment generating functions is

\[
G(t_1, t_2) = e^{1M' t_1 A_1 (I - t_1 A_1)^{-1} M} |I - t_1 A_1|^{-1} \\
\cdot e^{1M' t_2 A_2 (I - t_2 A_2)^{-1} M} |I - t_2 A_2|^{-1}.
\]

Assume \(A_1 A_2 = 0\). Then \((I - t_1 A_1 - t_2 A_2) = (I - t_1 A_1)(I - t_2 A_2)\)
and \(|I - t_1 A_1 - t_2 A_2| = |I - t_1 A_1| \cdot |I - t_2 A_2|\). Also

\((t_1 A_1 + t_2 A_2)(I - t_1 A_1 - t_2 A_2)^{-1} = t_1 A_1 (I - t_1 A_1)^{-1} + t_2 A_2 (I - t_2 A_2)^{-1},\)

for using the identity \(tA(I - tA)^{-1} = (I - tA)^{-1} - I,\) this becomes

\((I - t_2 A_2)^{-1}(I - t_1 A_1)^{-1} = (I - t_1 A_1)^{-1} + (I - t_2 A_2)^{-1} - I.\)

Multiplying both sides on the left by \((I - t_2 A_2)\) and on the right by \((I - t_1 A_1)\),
the identity follows. Thus the condition is sufficient.

It is apparent from the form of the moment generating functions that
a necessary condition for \(G(t_1, t_2) = G_Q(t_1) G_Q(t_2)\) is that \(|I - t_1 A_1 - t_2 A_2| =
|I - t_1 A_1| \cdot |I - t_2 A_2|\). However, it has been proved by Hotelling [3] and
Craig [2] that the condition \(A_1 A_2 = 0\) is necessary for this equality.

An extension can be made to correlated variates. Let \(X' = (x_1, \ldots, x_n)\)
be normally distributed with non-singular correlation matrix \(B\) and means
\(M' = (m_1, \ldots, m_n)\). Then there exists a non-singular transformation \(X = TZ,\)
such that the variates \(Z\) are independent and have unit variance. Thus
\(T^{-1} B T^{-1} = I, B = T T'\) and \(Q(X) = X' A' X = Z' T' A T Z\). Applying the theorem
proved above, a necessary and sufficient condition that \(Q(X)\) be distributed as
\(\chi^2\) is that \((T' A T) = T' A B A T T' = T' A T,\) or that \(ABA = A\). As before,
\(\chi^2 = Q(M)/2.\) In the same manner, a necessary and sufficient condition for
independence of \(Q_1(X)\) and \(Q_2(X)\) is that \((T' A_1 T)(T' A_2 T) = T' A_1 B A_2 T = 0,\)
or that \(A_1 B A_2 = 0,\)

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