DISTRIBUTION OF THE SUM OF ROOTS OF A DETERMINANTAL EQUATION UNDER A CERTAIN CONDITION

By D. N. Nanda

University of North Carolina

1. Summary. This paper is in continuation of the author's first two papers [1] and [2]. In this paper a method is described by which it is possible to derive the distribution of the sum of roots of a certain determinantal equation under the condition that $m = 0$. This condition implies, when the results are applied to canonical correlations, that the numbers of variates in the two sets differ by unity. The distributions for the sum of roots under this condition have been obtained for $l = 2, 3$ and 4 and are given in this paper. This paper also derives the moments of these distributions.

2. Introduction. The reader should refer to the first two papers of this series [1] and [2] for detailed explanation of the preliminaries essential for this paper.

The distribution of any root of the determinantal equation, specified by its rank when the roots are arranged in a descending order of magnitude, was derived by the author [1]. The distribution of the largest root was expressed as

$$(1) \qquad P_r(\theta_1 \leq x) = C(l, m, n)F_{l,m,n}(x) = \text{const.}\ (0, l, l-1, \cdots, 1, x; m, n).$$

3. Method. Putting $\theta_i = \rho_i/n$ in $R(l, m, n)$ as given in [1] and allowing n to tend to infinity, the distribution density reduces to

$$R(l, m) = \text{const.}\ \Pi\rho_i^m \prod_{i<j} (\rho_i - \rho_j)e^{-\Sigma\rho_i} \qquad (0 < \rho_l < \rho_{l-1} < \cdots < \rho_1 < \infty),$$

where the constant is independent of n, by [2]. If we replace x by x/n in the right-hand side of (1) and allow n to tend to infinity, then the resulting function $G_{l,m}(x)$ is independent of n and it can be shown by comparing the two methods A and B in [2], that

$$(2) \qquad \int_{0<\rho_l<\rho_{l-1}<\cdots<\rho_1<x} R(l, m)\Pi\, d\rho_i = G_{l,m}(x).$$

This is a constant multiple of

$$(3) \qquad \phi(x, m) = \int_{0<\rho_l<\rho_{l-1}<\cdots<\rho_1<x} \Pi\rho_i^m \prod_{i<j} (\rho_i - \rho_j)e^{-\Sigma\rho_i}\, \Pi\, d\rho_i$$
$$= \text{const.}\ x^{l+lm+l(l-1)/2}\,\theta(x, m).$$

Putting $\rho_i = xy_i$, we have

$$(4) \qquad \int_{0<y_l<y_{l-1}<\cdots<y_1<1} \Pi y_i^m \prod_{i<j} (y_i - y_j)e^{-x\Sigma y_i}\, \Pi\, dy_i = \text{const.}\ \theta(x, m).$$

The left-hand side is proportional to the moment generating function for the sum of roots when $n = 0$.

Let $y_1 = 1 - \theta_l$, $y_2 = 1 - \theta_{l-1}$, $\cdots$, $y_l = 1 - \theta_1$; then (4) gives

$$(5) \quad \int_{0<\theta_l<\theta_{l-1}<\cdots<\theta_1<1} \Pi(1 - \theta_i)^m \prod_{i<j} (\theta_i - \theta_j) e^{-lx+x\Sigma\theta_i} \Pi\, d\theta_i = \text{const.}\, \theta(x, m).$$

Let m be changed to n and both sides be multiplied by e^{lx}, then we get

$$(6) \quad \int_{0<\theta_l<\theta_{l-1}<\cdots<\theta_1<1} \Pi(1 - \theta_i)^n \prod_{i<j} (\theta_i - \theta_j) e^{x\Sigma\theta_i} \Pi d\theta_i = \text{const.}\, e^{lx} \theta(x, n).$$

The left-hand side of (6) is the moment generating function for the sum of roots when $m = 0$.

The method for obtaining the probability distributions is described in detail for each of the cases $l = 2, 3$, in the following sections.

It may, however, be added here that the condition $m = 0$, implies that $| p - q | = 1$ in the case of canonical correlations. It also implies, in generalized analysis of variance, that if we have K samples and measurements are made on p characters then $K - 1$ and p should differ by unity. Thus the distribution is given for 5 samples and 3 characters when $l = 3$ $(p = 3)$.

4. Distribution of the sum of roots when m = 0.

(*a*) $l = 2$. The value of $G_{2,m}(x)$ has been given in [2] as

$$(7) \qquad G_{2,m}(x) = k(2, m) \left[2 \int_0^x u^{2m+1} e^{-2u}\, du - x^{m+1} e^{-x} \int_0^x u^m e^{-u}\, du \right],$$

where $K(2, m) = 2^{2m+1}/\Gamma(2m + 2)$. Then in the notation just given

$$\phi(x, m) = 2 \int_0^x u^{2m+1} e^{-2u}\, du - x^{m+1} e^{-x} \int_0^x u^m e^{-u}\, du.$$

Replacing u by xu, we get

$$\begin{aligned} \phi(x, m) &= 2x^{2m+2} \int_0^1 u^{2m+1} e^{-2xu}\, du - x^{2m+2} e^{-x} \int_0^1 u^m e^{-xu}\, du \\ (8) \qquad &= \frac{x^{2m+2}}{m + 1} \int_0^1 e^{-2xu}\, d(u^{2m+2}) - \frac{x^{2m+2} e^{-x}}{m + 1} \int_0^1 e^{-xu}\, d(u^{m+1}) \\ &= \frac{x^{2m+3}}{m + 1} \left[2 \int_0^1 u^{2m+2} e^{-2xu}\, du - e^{-x} \int_0^1 u^{m+1} e^{-xu}\, du \right]. \end{aligned}$$

Hence

$$\theta(x, m) = \text{const.} \left[2 \int_0^1 u^{2m+2} e^{-2xu}\, du - e^{-x} \int_0^1 u^{m+1} e^{-xu}\, du \right],$$

and according to (6),

$$\int_{0<\theta_2<\theta_1<1} \Pi(1-\theta_i)^n(\theta_1-\theta_2)e^{x\Sigma\theta_i}\,d\theta_1\,d\theta_2$$

$$(9)\qquad = \text{const. } e^{2x}\left[2\int_0^1 u^{2n+2}e^{-2xu}\,du - e^{-x}\int_0^1 u^{n+1}e^{-xu}\,du\right]$$

$$= \text{const.}\left[2\int_0^1 (1-u)^{2n+2}e^{2xu}\,du - \int_0^1 (1-u)^{n+1}e^{xu}\,du\right],$$

by replacing u by $1-u$. Or,

$$E(e^{x\Sigma\theta_i}) = \text{const.}\left[2\int_0^1 (1-u)^{2n+2}e^{2xu}\,du - \int_0^1 (1-u)^{n+1}e^{xu}\,du\right].$$

The constant can be evaluated by putting $x = 0$.

Then let $P_r(\theta_1+\theta_2 \le Z) = \text{const.}\ [F_1(Z)+F_2(Z)]$, where $F_1(Z)$ and $F_2(Z)$ are cumulative distribution functions given by integrating the density $(1-u)^{2n+2}$ of $2u$ and $(1-u)^{n+1}$ of u, respectively. It is easily seen that

$$F_2(Z) = \int_0^Z (1-u)^{n+1}\,du = [1-(1-Z)^{n+2}]/(n+2) \qquad (Z \le 1).$$

Since $F_1(Z)$ is to be obtained from the density of $2u$, we may substitute $v = 2u$ and then integrate. Thus

$$F_1(Z) = 2\int_0^Z \left(1-\frac{v}{2}\right)^{2n+2} dv/2 = 2[1-(1-Z/2)^{2n+3}]/(2n+3) \quad (Z \le 2).$$

Hence the result for $l = 2$ is

$$P_r(\theta_1+\theta_2 \le Z) = 2(n+2)[1-(1-Z/2)^{2n+3}] - (2n+3)[1-(1-Z)^{n+2}] \qquad (0 \le Z \le 1),$$

$$= 2(n+2)[1-(1-Z/2)^{2n+3}] - (2n+3) \qquad (1 \le Z \le 2).$$

(*b*) $l = 3$. The value of $G_{3,m}(x)$ as given in [2] is changed as

$$G_{3,m}(x) = K(3,m)\left\{2\int_0^x u^{2m+3}e^{-2u}\,du\int_0^x u^m e^{-u}\,du - 2\int_0^x u^{m+1}e^{-u}\,du \cdot \int_0^x u^{2m+2}e^{-2u}\,du - \frac{x^{m+2}e^{-x}}{m+1}\left[2x^{2m+3}\int_0^1 u^{2m+2}e^{-2xu}\,du - x^{2m+3}e^{-x} \cdot \int_0^1 u^{m+1}e^{-xu}\,du\right]\right\},$$

using (8). $K(3, m)$ is a constant independent of n. Putting xu for u in only the first two terms of the right-hand side of the above equation, we get

$$G_{3,m}(x) = k(3, m)x^{3m+5}\left\{2\int_0^1 u^{2m+3}e^{-2xu}\,du\int_0^1 u^m e^{-xu}\,du\right.$$

$$(10)\qquad -2\int_0^1 u^{2m+2}e^{-2xu}\,du\int_0^1 u^{m+1}e^{-xu}\,du - \frac{2e^{-x}}{m+1}\int_0^1 u^{2m+2}e^{-2xu}\,du$$

$$\left. + \frac{e^{-2x}}{m+1}\int_0^1 u^{m+1}e^{-xu}\,du\right\}.$$

By integrating by parts we get x^{3m+6} as a common factor on the right-hand side of the above equation. Then according to (5) and (6) we have

$$\int_{0<y_3<y_2<y_1<1} \Pi y_i^m \prod_{i<j}(y_i - y_j)e^{-x\Sigma y_i}\,\Pi\,dy_i = \text{const.}\left\{2(m+2)\right.$$

$$\int_0^1 u^{2m+3}e^{-2xu}\,du\int_0^1 u^{m+1}e^{-xu}\,du + 2(2m+3)e^{-x}\int_0^1 u^{2m+4}e^{-2xu}\,du$$

$$\left. - 4(m+2)e^{-x}\int_0^1 u^{2m+3}e^{-2xu}\,du + e^{-2x}\int_0^1 u^{m+2}e^{-xu}\,du\right\}.$$

Putting $y_1 = 1 - \theta_3$, $y_2 = 1 - \theta_2$, $y_3 = 1 - \theta_1$ and, changing m to n and multiplying with e^{3x} we get

$$\int_{0<\theta_3<\theta_2<\theta_1<1} \Pi(1-\theta_i)^n \prod_{i<j}(\theta_i - \theta_j)e^{x\Sigma\theta_i}\,\Pi\,d\theta_i$$

$$(11)\qquad = \text{const.}\left\{2(n+2)\int_0^1 u^{2n+3}e^{2x(1-u)}\,du\int_0^1 u^{n+1}e^{x(1-u)}\,du\right.$$

$$+ 2(2n+3)\int_0^1 u^{2n+4}e^{2x(1-u)}\,du - 4(n+2)\int_0^1 u^{2n+3}e^{2x(1-u)}\,du$$

$$\left. + \int_0^1 u^{n+2}e^{x(1-u)}\,du\right\}.$$

Thus we have

$$P_r(\theta_1 + \theta_2 + \theta_3 \le Z) = \text{const.}\{F_1(Z) + F_2(Z) + F_3(Z) + F_4(Z)\},$$

where $F_1(Z)$, $F_2(Z)$, $F_3(Z)$ and $F_4(Z)$ are the contributions to the cumulative distribution by the four terms of the right-hand side of the following equation

$$E(e^{x\Sigma\theta_i}) = \text{const.}\left\{2(n+2)\int_0^1 (1-u)^{2n+3}e^{2xu}\,du\int_0^1 (1-u)^{n+1}e^{xu}\,du\right.$$

$$+ 2(2n+3)\int_0^1 (1-u)^{2n+4}e^{2xu}\,du - 4(n+2)\int_0^1 (1-u)^{2n+3}e^{2xu}\,du$$

$$\left. + \int_0^1 (1-u)^{n+2}e^{xu}\,du\right\},$$

where const. $= [(n+2)(n+3)(2n+5)]$. Proceeding according to the method given in (*a*) we have

$$(12)\qquad F_4(Z) = [1 - (1-Z)^{n+3}]/(n+3) \qquad (0 \le Z \le 1),$$

(13) $$F_2(Z) = 2(2n + 3)[1 - (1 - Z/2)^{2n+5}]/(2n + 5) \qquad (0 \le Z \le 2),$$

(14) $$F_3(Z) = -4(n + 2)[1 - (1 - Z/2)^{2n+4}]/(2n + 4) \qquad (0 \le Z \le 2).$$

Let us now consider $F_1(Z)$, which is the contribution of the first term. Let y_1 and y_2 be distributed between 0 and 1 with densities $(1 - y_1)^{2n+3}$ and $(1 - y_2)^{n+1}$

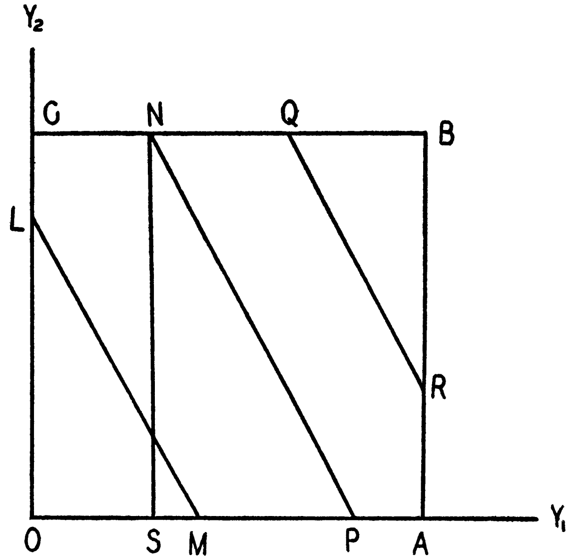

FIG. 1

respectively, then

$$F_1(Z) = 2(n + 2) \iint\limits_{2y_1+y_2 \le Z} (1 - y_1)^{2n+3}(1 - y_2)^{n+1}\, dy_1\, dy_2,$$

where Z goes from 0 to 3.

Let us consider the distribution over the unit square $OABC$, Fig. 1, then for $Z \le 1$, $Z \le 2$, and $Z \le 3$; we have to integrate over OLM, $OCNP$, and $OCQRA$, where LM, NP and QR are the three lines given by $2y_1 + y_2 \le Z$ according as $Z \le 1$, $Z \le 2$, and $Z \le 3$.

(i) The integration over OLM is given below

$$F_{1,1}(Z) = 2(2n + 2) \iint\limits_{2y_1+y_2 \le Z} (1 - y_1)^{2n+3}(1 - y_2)^{n+1}\, dy_1\, dy_2 \qquad \text{for } Z \le 1,$$

or

(15) $$F_{1,1}(Z) = 2\left\{\frac{1}{2n + 4}[1 - (1 - Z/2)^{2n+4}] - \lambda \cdot 2^{n+2}\left(\frac{3 - Z}{2}\right)^{3n+6}[I_{2/(3-Z)}(2n + 4, n + 3) - I_{(2-Z)/(3-Z)}(2n + 4, n + 3)]\right\},$$

where

$$\lambda = B(2n + 4, n + 3) = \int_0^1 y^{2n+3}(1 - y)^{n+2}\, dy$$

and

$$\lambda I_{2/(3-Z)} = \int_0^{2/(3-Z)} y^{2n+3}(1 - y)^{n+3}\, dy.$$

(ii) The integration over $OCNP$ is given below.

(16) $F_{1,2}(Z) = [1 - (1 - Z/2)^{2n+4}]/(n + 2)(2n + 4) - 2^{n+2}[(3 - Z)/2]^{(3n+6)}$
$\{B(2n + 4, n + 3) - \lambda I_{(2-Z)/(3-Z)}(2n + 4, n + 3)\}/(n + 2) \qquad (Z \leq 2).$

(iii) In order to integrate over $OCQRA$, we shall integrate over the unit area $OCBA$ and subtract from this the value obtained by integrating over QRB. Thus,

(17) $F_{1,3}(Z) = 1/(n + 2)(2n + 4) - 2^{n+2}[(3 - Z)/2]^{(3n+6)}$
$B(2n + 4, n + 3)/(n + 2).$

Hence the result for $l = 3$ can be expressed as

$$P_r(\theta_1 + \theta_2 + \theta_3 \leq Z) = \text{const.}\ \{F_{1,1}(Z) + F_2(Z) + F_3(Z) + F_4(Z)\}$$
$$= \text{const.}\ \{2(n + 2)\{[1 - (1 - Z/2)^{2n+4}]/(n + 2)(2n + 4)$$
$$-\lambda\cdot 2^{n+2}[^{(3-Z)}/2]^{3n+6}[I_{2/(3-Z)}(2n + 4, n + 3)$$
$$- I_{(2-Z)/(3-Z)}(2n + 4, n + 3)]/(n + 2)\}$$
$$+ 2(2n + 3)[1 - (1 - Z/2)^{2n+5}]/(2n + 5) - 2[1 - (1 - Z/2)^{2n+4}]$$
$$+ [1 - (1 - Z)^{n+3}]/(n + 3)\} \qquad (0 \leq Z \leq 1),$$

and

$$= \text{const.}\ \{F_{1,2}(Z) + F_2(Z) + F_3(Z) + F_4(1)\}$$
$$= \text{const.}\left[2(n + 2)\left[[1 - (1 - Z/2)^{2n+4}]/(n + 2)(2n + 4)\right.\right.$$
$$\left.- 2^{n+2}\left(\frac{3 - Z}{2}\right)^{3n+6}[B(2n + 4, n + 3) - \lambda I_{(2-Z)/(3-Z)}(2n + 4, n + 3)]/(n + 2)\right]$$
$$\left.+ 2(2n + 3)[1 - (1 - Z/2)^{2n+5}]/(2n + 5) - 2[1 - (1 - Z/2)^{2n+4}] + 1/n+3\right]$$
$$(1 \leq Z \leq 2),$$

and

$$= \text{const.} \{F_{1,3}(Z) + F_2(2) + F_3(2) + F_4(1)\}$$

$$= \text{const.} \left\{ 2(n+2) \left\{ 1/(n+2)(2n+4) - 2^{n+2} \left(\frac{3-Z}{2}\right)^{3n+6} \right.\right.$$

$$\left.\left. B(2n+4, n+3)/(n+2) \right\} + 2(2n+3)/(2n+5) - 2 + 1/(n+3) \right\}$$

$$(2 \leq Z \leq 3),$$

where const. $= (n+2)(n+3)(2n+5)$ and $\lambda = B(2n+4, n+3)$.

The exact distribution is obtained for $l = 4$ by the similar method. The final results are available with the author and are not given here due to lack of space.

The method given in the above sections can be used to find the distribution of the sum of roots of a determinantal equation of any order under the condition $m = 0$.

5. Moments of the distributions. The moments can be obtained by expanding the right-hand side of (6) in terms of x and then collecting the coefficients of x. The moments for $l = 2$ have been derived here and the method is illustrated below:

(*a*) $l = 2$. Equation (9) gives

$$\int_{0<\theta_2<\theta_1<1} \Pi(1-\theta_i)^n (\theta_1 - \theta_2) e^{x\Sigma\theta_i} \Pi\, d\theta_i = \text{const.} \left\{ 2 \int_0^1 (1-u)^{2n+2} e^{2xu}\, du \right.$$

$$\left. - \int_0^1 (1-u)^{n+1} e^{xu}\, du \right\} = \text{const.} \left\{ 2 \int_0^1 (1-u)^{2n+2} \sum_{t=0}^{\infty} \frac{(2xu)^t}{t!} \right.$$

$$\left. - \int_0^1 (1-u)^{n+1} \sum_{t=0}^{\infty} \frac{(xu)^t}{t!} \right\} = \text{const.} \left\{ 2 \sum_{t=0}^{\infty} \frac{(2x)^t}{t!} \frac{\Gamma(t+1)\Gamma(2n+3)}{\Gamma(2n+t+4)} \right.$$

$$\left. - \sum_{t=0}^{\infty} \frac{x^t}{t!} \frac{\Gamma(t+1)\Gamma(n+2)}{\Gamma(n+t+3)} \right\} = \text{const.} \left\{ \frac{2}{2n+3} \left[1 + \frac{2x}{2n+4} \right.\right.$$

$$\left. + \frac{(2x)^2}{(2n+4)(2n+5)} + \frac{(2x)^3}{(2n+4)(2n+5)(2n+6)} + \cdots \right]$$

$$- \frac{1}{(n+2)} \left[1 + \frac{x}{n+3} + \frac{x^2}{(n+3)(n+4)} \right.$$

$$\left.\left. + \frac{x^3}{(n+3)(n+4)(n+5)} + \cdots \right] \right\}.$$

Thus

$$E(e^{x\Sigma\theta_i}) = \left\{ 1 + \frac{x}{1!} \cdot \frac{3}{(n+3)} + \frac{x^2}{2!} \frac{12(n+2)(4n+11)}{(n+3)(n+4)(2n+4)(2n+5)} \right.$$

$$\left. + \frac{x^3}{3!} \frac{120(n+2)(n+3)(4n+13)}{(2n+4)(2n+5)(2n+6)(n+3)(n+4)(n+5)} + \cdots \right\}.$$

Hence

$$\mu_1' = 3/(n + 3),$$
$$\mu_2' = 6(4n + 11)/(n + 3)(n + 4)(2n + 5)$$

and

$$\mu_3' = 30(4n + 13)/(n + 3)(n + 4)(n + 5)(2n + 5).$$

The moments for $l = 3$ and 4 can be obtained in a similar way.

Acknowledgements. The problem was suggested to me by Dr. P. L. Hsu. I take this opportunity to express my gratitude to Dr. P. L. Hsu for guiding me in this research. I am also indebted to Dr. Harold Hotelling for help and suggestions in the work.

REFERENCES

[1] D. N. NANDA, "Distribution of a root of a determinantal equation," *Annals of Math. Stat.*, Vol. 19 (1948), pp. 47–57.
[2] D. N. NANDA, "Limiting distribution of a root of a determinantal equation," *Annals of Math. Stat.*, Vol. 19 (1948), pp. 340–350.