ON UNIFORMLY CONSISTENT TESTS

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1. Introduction. If we wish to decide on the true distribution of a random variable known to be distributed according to one or the other of two given distributions F_0 and F_1, then, no matter how small a bound is given in advance, it is always possible to devise a test based on a sufficiently large number of independent observations for which the probabilities of erroneous decisions are smaller than the previously assigned bound. A sequence of tests for which the corresponding probabilities of errors tend to zero has been called consistent [1].[1]

Let us suppose now that all we know about the true distribution of some random variable is that it belongs to one of two given families of distributions and it is desired to decide which of the two it belongs to; i.e., we have to test a composite hypothesis. It may again be possible to construct a sequence of tests $\{T_j\}$, $j = 1, 2, \cdots$, such that for any $\epsilon > 0$ there exists an index N such that for $j > N$ the probabilities of errors corresponding to T_j are smaller than ϵ. The sequence $\{T_j\}$ may then be called uniformly consistent. Conditions under which uniformly consistent tests exist have been given by von Mises [5], and by Wald [2], [3], [4], as implied, for example, by his proof of the uniform consistency of the likelihood ratio test. In this paper a different set of conditions is given which do not restrict in any way the nature of the distribution functions considered. It is also shown that the conditions to be described are satisfied in a large class of cases occurring in practical statistics.

Since the results we are to prove have their counterpart in abstract measure theory we shall take advantage of that method. The reader will have no difficulty in establishing the correspondence between the statistical and measure theoretical formulation.

Notations. Let X be an arbitrary set and $\mathfrak{B}$ a Borel field of subsets B of X. Let $\mathfrak{M}(\mathfrak{B})$ be the family of all probability measures $m(B)$ defined on $\mathfrak{B}$, i.e., the family of all countably additive nonnegative set functions defined on $\mathfrak{B}$ for which $m(X) = 1$. Hereafter a "measure" will denote an element of $\mathfrak{M}(\mathfrak{B})$ and a "set of measures" a subset of $\mathfrak{M}(\mathfrak{B})$. For any positive integer k, let X^k be the kth direct product of X by itself, $\mathfrak{B}^k$ the kth direct product of $\mathfrak{B}$ by itself, $\mathfrak{C}^{\cdot k}$ the field consisting of all finite sums of sets of $\mathfrak{B}^k$ and $\mathfrak{B}^{\cdot k}$ the smallest σ-field containing $\mathfrak{C}^{\cdot k}$. For any measure m on $\mathfrak{B}$, we define m^k in the usual way as the unique measure defined on $\mathfrak{B}^{\cdot k}$ for which

$$m^k\left(\sum_{i=1}^{l} B_{i1}\cdot B_{i2}\cdot \cdots \cdot B_{ik}\right) = \sum_{i=1}^{l} m(B_{i1})m(B_{i2}) \cdots m(B_{ik})$$

for any disjoint system $B_{ij} \varepsilon \mathfrak{B}$, $i = 1, 2, \cdots, l$; $j = 1, 2, \cdots, k$, where l is an arbitrary positive integer.

[1] This is a slightly modified form of the definition in [1].

2. A known lemma. The main result will be established by a generalization of the following well known lemma:

LEMMA 1 (BERNOULLI). *Let $M = \{m\}$ and $M' = \{m'\}$ be two disjoint sets of measures. If there exists a set A in $\mathfrak{B}$ and a $\delta > 0$ such that*

$$| m(A) - m'(A) | > 2\delta$$

for all m in M and all m' in M', then, for any $\epsilon > 0$ given in advance, there exists an integer k and a set E in $\mathfrak{F}^{\cdot k}$ such that

$$m^k(E) < \epsilon \qquad \textit{for all } m \textit{ in } M$$

and

$$m'^k(E) > 1 - \epsilon \qquad \textit{for all } m' \textit{ in } M'.$$

This is an almost immediate consequence of Bernoulli's theorem, but for sake of completeness I include a proof.

PROOF. For any two integers n and r, $0 \leq r \leq n$, let $R(n, r)$ be the union of all regions in X^n defined by restricting r of the first n coordinates to A, and the remaining $n - r$ to $(X - A)$. For any fixed n, let for any measure μ

$$S_\mu^n = \bigcup_{\{r \,|\, |\frac{r}{n} - \mu(A)| \geq \delta\}} R(n, r).$$

(Here $\{t \mid T\}$ means the set of all t's which satisfy relation T, as customary.) Let $\epsilon > 0$ be given. By Bernoulli's theorem, there exists an integer $n(\epsilon)$ such that

$$\mu^n(S_\mu^n) = \sum_{\{r \,|\, |\frac{r}{n} - \mu(A)| \geq \delta\}} \binom{n}{r} [\mu(A)]^r [1 - \mu(A)]^{n-r} < \epsilon.$$

Let $E = \bigcap_{m \in M} S_m^n$. Since for any fixed n there are only a finite number of different S_m^n, E is in $\mathfrak{F}^{\cdot n}$ and

$$m^n(E) < \epsilon$$

for all m in M. Since for any m in M and m' in M',

$$\left| \frac{r}{n} - m(A) \right| \geq 2\delta - \left| m'(A) - \frac{r}{n} \right|,$$

we have $| (r/n) - m(A) | > \delta$ for all r satisfying $| m'(A) - (r/n) | < \delta$ for some m' in M'. Hence if x is in $(X^n - S_{m'}^n)$ for some m' in M', then x is in E and

$$m'^n(E) > m'^n(X^n - S_{m'}^n) > 1 - \epsilon$$

for all m' in M'.

In the special case $M = m$, $M' = m'$, this proves the statement in the introduction concerning simple hypotheses.

3. The main results.

DEFINITION. *Let $M = \{m\}$ and $M' = \{m'\}$ be two disjoint sets of measures. We shall say that they satisfy "Condition 1" if the following holds: M is the union* *of a finite number of its subsets M_i, $i = 1, 2, \cdots, k$, such that for every i there exist*

(i) *a covering of M' by a finite number of its subsets M'_{ij}, $j = 1, 2, \cdots, h_i$,*
(ii) *a sequence of sets A_{ij} in $\mathfrak{B}$, $j = 1, 2, \cdots, h_i$, and*
(iii) *a $\delta > 0$ such that $| m(A_{ij}) - m'(A_{ij}) | > \delta$ for every m in M_i and every m' in M'_{ij}, $j = 1, 2, \cdots, h_i$; $i = 1, 2, \cdots, k$.*

Condition 1 is satisfied for instance if both M and M' contain only a finite number of measures.

LEMMA 2. *Let $M = \{m\}$ and $M' = \{m'\}$ be two disjoint sets of measures and assume that they satisfy Condition 1. Then for every $\epsilon > 0$, there exist an integer $n(\epsilon)$ and a set E in $\mathfrak{E}^{\cdot n}$ such that $m^n(E) < \epsilon$ for all m in M and $m'^n(E) > 1 - \epsilon$ for all m' in M'.*

PROOF. Assume first that $k = 1$. Then $M_i = M_1 = M$, and we can put $M'_{1j} = M'_j$, $h_1 = h$, $A_{ij} = A_j$. Condition 1 then states that $| m(A_j) - m'(A_j)| > \delta$ for every m in M and m' in M', $j = 1, 2, \cdots, h$. By Lemma 1, for any $\epsilon > 0$ there exists n_j and E_j in $\mathfrak{E}^{\cdot n_j}$ such that $m^{n_j}(E_j) < \epsilon/h$ for all m in M and $m'^{n_j}(E_j) > 1 - \epsilon/h$ for all m' in M'_j. Let $n = \max n_j$ and

$$E = \bigcup_{j=1}^{h} E_j \cdot X^{n-n_j}.$$

Then E is in $\mathfrak{E}^{\cdot n}$ and

$$m^n(E) \leq \sum_{j=1}^{h} m^n(E_j \cdot X^{n-n_j}) = \sum_{j=1}^{h} m^{n_j}(E_j) < \epsilon$$

for every m in M, and if m' is in any fixed M'_j,

$$m'^n(E) > m'^{n_j}(E_j) > 1 - \frac{\epsilon}{h} \geq 1 - \epsilon,$$

so that

$$m'^n(E) > 1 - \epsilon$$

for all m' in M'.

Now if $k > 1$, let us choose some $\bar{\epsilon}$, $0 < \bar{\epsilon} < \frac{1}{2}$, and apply the above argument to each M_i. We get $m^{n_i}(E_i) < \bar{\epsilon}$ for all m in M_i and $m'^{n_i}(E_i) > 1 - \bar{\epsilon}$ for all m' in M'. Hence

$$| m^{n_i}(E_i) - m'^{n_i}(E_i)| > 1 - 2\bar{\epsilon} > 0,$$

so that Condition 1 is satisfied with $k = 1$ and with the set $\{m'^{(\max n_i)}\}$ taking the place of M and the set $\{m^{(\max n_i)}\}$ taking the place of M'. It is easy to see that also in this case E still belongs to the field $\mathfrak{E}^{\cdot n}$.

If we do not require that the set E in the conclusion of Lemma 2 belong to $\mathfrak{E}^{\cdot n}$ but only that it belong to $\mathfrak{B}^{\cdot n}$, we can relax Condition 1 in the following obvious way:

THEOREM 1. *In order that two disjoint sets of measures $M = \{m\}$ and $M' = \{m'\}$ be such that for every $\epsilon > 0$ there be an integer n and a set B in $\mathfrak{B}^{\cdot n}$ for which*

$m^{\nu}(B) < \epsilon$ for every m in M and $m'^{\nu}(B) > 1 - \epsilon$ for every m' in M', it is necessary and sufficient that for some integer ν the sets $\{m^{\nu}\}$ and $\{m'^{\nu}\}$ satisfy Condition 1.

THEOREM 2. *Let $M = \{m_\theta\}$, $M' = \{m'_\tau\}$ be two disjoint sets of measures, $a \leq \theta \leq b$, $a' \leq \tau \leq b'$, where $[a, b]$ and $[a', b']$ are two disjoint, closed intervals of some finite-dimensional Euclidean space, and assume that, for each B in $\mathfrak{B}$, $m_\theta(B)$ and $m'_\tau(B)$ are continuous functions of θ and τ, respectively. Then for any $\epsilon > 0$ given in advance, there exist an integer $n(\epsilon)$ and a set E in $\mathfrak{E}^n$ such that $m_\theta^n(E) < \epsilon$ for all θ in $[a, b]$ and $m_\tau'^n(E) > 1 - \epsilon$ for all τ in $[a', b']$.*

PROOF. It is sufficient to prove that M and M' satisfy Condition 1. For any θ in $[a, b]$ and any τ in $[a', b']$, let $B_{\theta\tau}$ denote a set in $\mathfrak{B}$ for which

$$| m_\theta(B_{\theta\tau}) - m'_\tau(B_{\theta\tau})| > \epsilon_{\theta\tau} > 0. \tag{1}$$

(This is obviously possible.)

Let us now hold θ fixed. Because of the continuity of m'_τ, for every τ there exists a $\delta_{\theta\tau} > 0$ such that whenever $| \bar{\tau} - \tau | < \delta_{\theta\tau}$ then

$$| m'_{\bar{\tau}}(B_{\theta\tau}) - m'_\tau(B_{\theta\tau}) | < \frac{\epsilon_{\theta\tau}}{3}. \tag{2}$$

Since $[a', b']$ is compact, it can be covered for each fixed θ by a finite subset of the open intervals $I_{\theta\tau} = (-\delta_{\theta\tau} + \tau, \tau + \delta_{\theta\tau})$, say $I_{\theta 1}, I_{\theta 2}, \cdots, I_{\theta h(\theta)}$, with midpoints $\tau_{\theta 1}, \tau_{\theta 2}, \cdots, \tau_{\theta h(\theta)}$. Denote the values of m'_τ, $h_{\theta\tau}$, $B_{\theta\tau}$, $\epsilon_{\theta\tau}$, $\delta_{\theta\tau}$ at $\tau = \tau_{\theta j}$ by $m'_{\theta j}$, $h_{\theta j}$, $B_{\theta j}$, $\epsilon_{\theta j}$, $\delta_{\theta j}$, respectively, for $j = 1, 2, \cdots, h(\theta)$.

Since m_θ is continuous in θ for all B, there exists a positive number ρ_θ such that whenever $| \bar{\theta} - \theta | < \rho_\theta$ then simultaneously for $j = 1, 2, \cdots, h(\theta)$

$$| m_\theta(B_{\theta j}) - m_{\bar{\theta}}(B_{\theta j})| < \tfrac{1}{3} \min_j \epsilon_{\theta j}, \tag{3}$$

and since $[a, b]$ is compact, it can be covered by a finite subset of the open intervals $L_\theta = (-\rho_\theta + \theta, \theta + \rho_\theta)$, say $L_i = (-\rho_i + \theta_i, \theta_i + \rho_i)$, $i = 1, 2, \cdots, k$. Let us denote the values of $\tau_{\theta j}$, $B_{\theta j}$, $h(\theta)$, $\epsilon_{\theta j}$, $\delta_{\theta j}$, $I_{\theta j}$ at $\theta = \theta_i$ by τ_{ij}, B_{ij}, h_i, ϵ_{ij}, δ_{ij}, I_{ij}, respectively. Then the sets $M_i = \{m_\theta \mid \theta \text{ in } L_i\}$, $i = 1, 2, \cdots, k$, cover M, and for each i the sets $| M'_{ij} = \{m'_\tau \mid \tau \text{ in } I_{ij}\}$, $j = 1, 2, \cdots, h_i$, cover M'. Furthermore it follows from (1), (2), and (3) that as long as θ is in M_i and τ in M'_{ij},

$$\begin{aligned} | m_\theta(B_{ij}) - m'_\tau(B_{ij})| &\geq | m_{\theta_i}(B_{ij}) - m'_{\tau_{ij}}(B_{ij})| - | m_{\theta_i}(B_{ij}) - m_\theta(B_{ij})| \\ &\quad - | m'_\tau(B_{ij}) - m'_{\tau_{ij}}(B_{ij})| > \epsilon_{ij} - \tfrac{1}{3} \min_j \epsilon_{ij} - \tfrac{1}{3} \epsilon_{ij} > \eta > 0, \\ &\qquad i = 1, 2, \cdots, k; \quad j = 1, 2, \cdots, h_i, \end{aligned}$$

as we wanted to prove.

In the statistical terminology of the introduction, Theorem 2 may be restated as follows: Let H_0 be the hypothesis that the unknown distribution of some random variable belongs to a set of distributions M, and H_1 that it belongs to another, M', where M and M' satisfy the assumptions of Theorem 2. Then there exists a uniformly consistent sequence of tests for testing H_0 against H_1.

4. An example. Let $F(t) = (1/\sqrt{2\pi})\, e^{-\frac{1}{2}(x-t)^2}$, $H_0 = F(0)$, $H_1 = \{F(t), 1 \leq t \leq 2\}$ Let

$$R_{n,i} = \left\{(x_1, x_2, \cdots, x_n) \,\middle|\, \frac{x_1 + x_2 + \cdots + x_n}{n} > c_i\right\},$$

$$i = 1, 2, \cdots ; n = 1, 2, \cdots,$$

where c_i is determined so that $P(R_{n,i} \mid 0) = 1/i$, $P(S \mid t)$ denoting the probability of the region S when t is the true mean. Thus $R_{n,i}$ is the uniformly most powerful region of size $1/i$ in n-dimensional sample space for testing H_0 against H_1. The regions $R_{n,i}$ define a uniformly consistent test. A proof avoiding all computation is based on Theorem 2 as follows. Let $\epsilon > 0$ be given; find i such that $1/i < \epsilon$. By Theorem 2, there exists an N and a Borel set B in the N-dimensional sample space such that $P(B \mid 0) < 1/i$ and $P(B \mid t) > 1 - 1/i$ for $1 \leq t \leq 2$. Let W be a region in N-dimensional sample space covering B and such that $P(W \mid 0) = 1/i$. Then $P(W \mid 0) = P(R_{n,i} \mid 0) = 1/i < \epsilon$, and by the definition of $R_{N,i}$

$$P(R_{N,i} \mid t) \geq P(W \mid t) \geq P(B \mid t) > 1 - 1/i > 1 - \epsilon, \qquad 1 \leq t \leq 2.$$

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