DESIGNS FOR TWO-WAY ELIMINATION OF HETEROGENEITY

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1. Introduction and summary. Sometimes in a design the position within the block is important as a source of variation, and the experiment gains in efficiency by eliminating the positional effect. The classical example is due to Youden in his studies on the tobacco mosaic virus [1]. He found that the response to treatments also depends on the position of the leaf on the plant. If the number of leaves is sufficient so that every treatment can be applied to one leaf of a tree, then we get an ordinary Latin square, in which the trees are columns and the leaves belonging to the same position constitute the rows. But if the number of treatments is larger than the number of leaf positions available, then we must have incomplete columns. Youden used a design in which the columns constituted a balanced incomplete block design, whereas the rows were complete. These designs are known as Youden's squares, and can be used when two-way elimination of heterogeneity is desired.

In Fisher and Yates statistical tables [2] balanced incomplete block designs in which the number of blocks b is equal to the number of treatments v have been used to obtain Youden's squares, and the authors state that "in all cases of practical importance" it has been found possible to convert balanced incomplete blocks of the above kind to a Youden's square by so ordering the varieties in the blocks that each variety occurs once in each position. F. W. Levi noted ([3], p. 6) that this reordering can always be done, in virtue of a theorem given by Konig [4] which states that an even regular graph of degree m is the product of m regular graphs of degree 1. Smith and Hartley [5] give a practical procedure for converting balanced incomplete blocks with $b = v$ into Youden's squares.

In this paper I have considered some general classes of designs for two-way elimination of heterogeneity. In Section 3 balanced incomplete block designs for which $b = mv$ have been used to obtain two-way designs in which each treatment occurs in a given position m times. The case $m = 1$ gives Youden's squares. In Section 4 it has been shown that balanced incomplete block designs for which b is not an integral multiple of v can be used to obtain designs for two-way elimination of heterogeneity in which there are two accuracies (i.e., some pairs of treatments are compared with one accuracy, while other pairs are compared with a different accuracy) as in the case of lattice designs for one-way elimination of heterogeneity. In Sections 5 and 6 partially balanced designs have been used to obtain two-way designs with two accuracies. In every case the method of analysis and tables of actual designs have been given.

2. Notation and preliminaries. Consider a two-way design with k rows and b columns. Let there be v treatments altogether, and let n_{ij} denote the number of times the treatment i occurs in row j, and n'_{ic} the number of times it occurs in the column c. If y_{jc} is the yield for the jth row and cth column, the mathematical model assumed will be

$$y_{jc} = g + t_i + b_j + p_c + e_{jc}, \tag{2.0}$$

where t_i is the effect of the treatment i occurring in the row j and column c, b_j and p_c are the effects of the jth row and cth column, respectively, and e_{jc} is a random variable which is distributed $N(0, \sigma^2)$ independently for each value of j and c.

Let T_i, B_j and B'_c denote respectively the totals of the yields corresponding to the treatment i, row j and column c. Put

$$Q_i = T_i - \frac{1}{b}\Sigma_j n_{ij} B_j - \frac{1}{k}\Sigma_c n'_{ic} B'_c + \frac{r_i G}{bk}, \tag{2.1}$$

where G is the grand total of all the yields and r_i is the number of replications of the ith treatment. Q_i is called the adjusted yield of the ith treatment.

Let us set

$$c_{ii} = r_i\left(1 + \frac{r_i}{bk}\right) - \frac{1}{b}\Sigma_j n_{ij}^2 - \frac{1}{k}\Sigma_c n'^2_{ic}, \tag{2.2}$$

$$c_{iu} = -\frac{1}{b}\Sigma_j n_{ij} n_{uj} - \frac{1}{k}\Sigma_c n'_{ic} n'_{uc} + \frac{r_i r_u}{bk}, \qquad i \neq u. \tag{2.3}$$

It can be easily shown that the rank of the matrix (c_{iu}) is at the most equal to $v - 1$. We shall suppose that the parameters entering in the design are such that rank (c_{iu}) is actually equal to $v - 1$. In this case the design is said to be connected. The best unbiased linear estimate of any contrast

$$l_1 t_1 + l_2 t_2 + \cdots + l_v t_v, \qquad \Sigma_i l_i = 0, \tag{2.4}$$

is obtained by solving the normal equations

$$c_{i1} t_1 + c_{i2} t_2 + \cdots + c_{iv} t_v = Q_i, \qquad i = 1, 2, \cdots, v, \tag{2.5}$$

and substituting the values in the contrast. The t's are determined up to an arbitrary constant, and may be made unique by using the constraint

$$t_1 + t_2 + \cdots + t_v = 0. \tag{2.6}$$

Let $\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_v$ be any solution of (2.4). Then the analysis of variance table for the design will be Table I. Detailed proofs of the facts stated in this section can be worked out along the lines indicated by Bose [6].

3. Designs with complete rows in which every treatment occurs in a row m times. Consider a two-way design in which the columns form a balanced incomplete block design with parameters v, b, r, k, λ, where v is the number of treatments, b is the number of blocks, k is the block size, r is the number of replications of each treatment, and λ is the number of times any two treatments occur together in the same column. Then

$$\Sigma_c n_{ic}'^2 = r_i = r, \qquad i = 1, 2, \cdots, v, \tag{3.0}$$

$$\Sigma_c n_{ic}' n_{uc}' = \lambda, \qquad i, u = 1, 2, \cdots, v; i \neq u. \tag{3.1}$$

Consider the matrix $N = (n_{ij})$ of v rows and k columns, n_{ij} being the number already defined. The matrix N is intrinsically associated with the positions of the treatments within the columns of the design, and depends on the parameters of the design only inasmuch as each column adds up to b and each row to

TABLE I

Analysis of variance for a two-way design

Source of variation	d.f.	Sum of squares	Mean square
Treatment contrasts eliminating rows and columns	$v - 1$	$S_t^2 = \Sigma_i \hat{t}_i Q_i$	$s_t^2 = \dfrac{S_t^2}{v-1}$
Row contrasts ignoring treatments	$k - 1$	$\frac{1}{b}\Sigma_j B_j^2 - \frac{G^2}{bk}$	
Column contrasts ignoring treatments	$b - 1$	$\frac{1}{k}\Sigma_c B_c'^2 - \frac{G^2}{bk}$	
Error	$(b - 1)(k - 1) - (v - 1)$	S_e^2(by subtraction)	$s_e^2 = \dfrac{S_e^2}{(b-1)(k-1)-(v-1)}$
Total	$bk - 1$	$\Sigma_{j,c} y_{jc}^2 - \frac{G^2}{bk}$	

$F = \frac{s_t^2}{s_e^2}$, d.f. $v - 1$, $(b - 1)(k - 1) - (v - 1)$.

r. Let b be an integral multiple of v, so that $b = mv$. Then $r = mk$. By suitable interchanges of treatments in the same column of the design, the matrix N can be so modified that

$$n_{ij} = m, \qquad i = 1, 2, \cdots, v; j = 1, 2, \cdots, k, \tag{3.2}$$

since the procedure of Smith and Hartley [5] can be easily generalized in the following manner to cover the case $m \neq 1$.

If $n_{ij} = m$ is not satisfied for all values of i, j, we define

$$\begin{aligned} m_{ij} &= m - n_{ij} && \text{if } m > n_{ij}, \\ &= 0 && \text{if } m \leq n_{ij}, \\ M &= \Sigma m_{ij}. \end{aligned}$$

Then, following the Smith-Hartley procedure, only slight modifications in the argument show that we can find an interchange or system of interchanges within columns which would reduce M by at least unity. Successive applications of this process give the desired result, since $M = 0$ implies $n_{ij} = m$ for all i, j.

Now we have

$$\Sigma_j n_{ij}^2 = km^2, \qquad i = 1, 2, \cdots, v, \tag{3.3}$$

$$\Sigma_j n_{ij} n_{uj} = km^2, \qquad i, u = 1, 2, \cdots, v; i \neq u. \tag{3.4}$$

Under the restraint (2.6) the normal equations (2.5) become

$$\left(r - \frac{r}{k} + \frac{\lambda}{k}\right) t_i = Q_i, \tag{3.5}$$

which are exactly the same as for balanced incomplete block designs (cf. Bose [6]). Hence $\hat{t}_i$, the estimate of t_i, is Q_i/rE, where $E = v\lambda/kr$, and the analysis of variance can be obtained by substituting this value of $\hat{t}_i$ in the table at the end of Section 2. Also

$$V(\hat{t}_i - \hat{t}_u) = 2\sigma^2/rE. \tag{3.6}$$

When a cyclic or multicyclic solution of a balanced incomplete block design is available the matrix N already obeys the condition (3.2), and an actual application of the Smith-Hartley process is unnecessary. Only the designs with $r \leq 10$ are practically important, and cyclic or multicyclic solutions for all but three of these designs are available in the tables of Fisher and Yates [2], and in a paper by Rao [7]. The solution for the three missing cases are given in Table II. The solution for the design 1 of Table II is obtained by modifying the corresponding solution by Bose [8], and the solutions for designs 2 and 3 are obtained from the corresponding solutions by Bhattacharya [9], [10]. The designs considered here may be called extended Youden's square designs when $m \geq 1$.

In Table II, instead of giving the design in the row-column form, it is convenient to give the blocks corresponding to the columns. The row position is then given by the position within the block. This convention will be adopted throughout the paper. In many cases it is possible to represent the designs compactly by developing a set of blocks from one block cyclically. The following convention will be adopted for this purpose. To develop the block $(a, b, \cdots, x)$ cyclically (mod g), we write down the set of g blocks $(a + t, b + t, \cdots, x + t)$, $t = 0, 1, 2, \cdots, g - 1$, and then reduce every number appearing in the blocks to lie between 1 and g (both inclusive) by subtracting g whenever a number appearing in the blocks exceeds g. In certain cases to each number between 1 and g there correspond m treatments instead of one. The treatments corresponding to c being denoted by $c_1, c_2, \cdots, c_m$. In this case in developing the blocks suffixes are left invariant. Thus by developing $(1_1, 5_2, 4_1)$ cyclically (mod 5), we get $(1_1, 5_2, 4_1)$, $(2_1, 1_2, 5_1)$, $(3_1, 2_2, 1_1)$, $(4_1, 3_2, 2_1)$, $(5_1, 4_2, 3_1)$.

Sometimes treatments are represented by compound symbols (a, b) with or without suffixes, and we have to develop a block (mod g_1, g_2). This can be done analogously. For example, by developing (mod 3, 3) the block

$$[(2, 1)_1, (1, 2)_1, (2, 2)_2, (1, 1)_2],$$

we get the nine blocks

$$[(2, 1)_1, (1, 2)_1, (2, 2)_2, (1, 1)_2], \quad [(3, 1)_1, (2, 2)_1, (3, 2)_2, (2, 1)_2],$$

$$[(1, 1)_1, (3, 2)_1, (1, 2)_2, (3, 1)_2], \quad [(2, 2)_1, (1, 3)_1, (2, 3)_2, (1, 2)_2],$$

$$[(3, 2)_1, (2, 3)_1, (3, 3)_2, (2, 2)_2], \quad [(1, 2)_1, (3, 3)_1, (1, 3)_2, (3, 2)_2],$$

$$[(2, 3)_1, (1, 1)_1, (2, 1)_2, (1, 3)_2], \quad [(3, 3)_1, (2, 1)_1, (3, 1)_2, (2, 3)_2],$$

$$[(1, 3)_1, (3, 1)_1, (1, 1)_2, (3, 3)_2].$$

TABLE II

Some extended Youden's square designs

Serial no.	Parameters: v, b, r, k, λ	Blocks	
1	10, 30, 9, 3, 2	$(5_2, 1_2, 2_2)$, $(1_1, 5_2, 4_1)$, $(2_1, 3_1, 5_2)$, $(1_1, 4_1, 2_2)$, $(2_2, 3_1, 2_1)$, $(5_2, 2_2, 5_1)$; other blocks are obtained by developing (mod 5), keeping the suffixes fixed.	
2	25, 25, 9, 9, 3	(5, 1, 23, 6, 20, 12, 17, 2, 11),	(18, 21, 5, 7, 10, 24, 3, 12, 1),
		(15, 2, 9, 10, 1, 21, 25, 17, 16),	(23, 22, 11, 9, 3, 18, 1, 16, 14),
		(24, 13, 2, 14, 7, 8, 22, 1, 17),	(8, 25, 20, 3, 6, 1, 13, 18, 15),
		(20, 4, 3, 17, 8, 10, 7, 23, 9),	(21, 8, 24, 11, 4, 6, 2, 9, 18),
		(14, 12, 13, 4, 17, 25, 21, 11, 3),	(3, 24, 17, 22, 15, 5, 16, 4, 6),
		(25, 5, 18, 20, 16, 14, 4, 7, 2),	(22, 19, 1, 5, 25, 11, 10, 8, 4),
		(19, 14, 6, 13, 9, 17, 18, 5, 10),	(1, 20, 15, 12, 19, 4, 9, 14, 24),
		(16, 7, 4, 1, 13, 23, 6, 21, 19),	(12, 16, 8, 23, 24, 9, 5, 25, 13),
		(9, 3, 25, 19, 22, 2, 12, 6, 7),	(7, 17, 12, 18, 11, 16, 15, 19, 8),
		(13, 11, 10, 16, 2, 3, 19, 24, 20),	(6, 10, 16, 8, 12, 20, 14, 22, 21),
		(2, 23, 21, 15, 14, 19, 8, 3, 5),	(10, 6, 14, 24, 23, 7, 11, 15, 25),
		(17, 18, 19, 25, 21, 22, 24, 20, 23),	(4, 15, 22, 2, 18, 13, 23, 10, 12),
		(11, 9, 7, 21, 5, 15, 20, 13, 22)	
3	31, 31, 10, 10, 3	(1, 2, 28, 15, 9, 11, 8, 16, 18, 4),	(2, 3, 22, 16, 10, 17, 9, 19, 5, 12),
		(3, 4, 23, 6, 17, 13, 10, 18, 11, 20),	(4, 5, 24, 18, 12, 21, 11, 14, 19, 7),
		(5, 6, 25, 19, 13, 8, 12, 20, 15, 1),	(6, 7, 26, 20, 14, 9, 16, 21, 2, 13),
		(7, 1, 27, 21, 8, 10, 14, 17, 3, 15),	(9, 12, 6, 1, 27, 18, 29, 26, 17, 24),
		(10, 13, 7, 2, 29, 25, 19, 27, 28, 18),	(11, 14, 1, 3, 19, 20, 26, 28, 29, 22),
		(12, 8, 2, 4, 20, 29, 23, 22, 21, 27),	(13, 9, 3, 5, 21, 15, 28, 23, 24, 29),
		(14, 10, 4, 24, 15, 16, 22, 29, 6, 25),	(8, 11, 5, 29, 16, 23, 25, 7, 26, 17),
		(15, 24, 20, 11, 2, 27, 5, 10, 30, 26),	(16, 25, 21, 12, 3, 28, 30, 11, 27, 6),
		(17, 26, 15, 13, 30, 22, 7, 12, 4, 28),	(18, 27, 16, 14, 5, 30, 1, 13, 22, 23),
		(19, 28, 17, 8, 6, 14, 2, 24, 23, 30),	(20, 22, 18, 9, 7, 24, 3, 30, 25, 8),
		(21, 23, 19, 30, 1, 26, 4, 15, 9, 10),	(24, 16, 8, 27, 26, 3, 31, 4, 13, 19),
		(25, 17, 9, 28, 31, 4, 27, 5, 20, 14),	(26, 18, 10, 22, 28, 31, 21, 6, 8, 5),
		(27, 19, 11, 23, 22, 6, 15, 9, 7, 31),	(28, 20, 12, 31, 23, 7, 24, 1, 10, 16),
		(22, 21, 13, 25, 24, 1, 17, 2, 31, 11),	(23, 15, 14, 26, 25, 2, 18, 31, 12, 3),
		(29, 30, 31, 7, 4, 5, 6, 3, 1, 2),	(30, 31, 29, 10, 11, 12, 13, 8, 14, 9),
		(31, 29, 30, 17, 18, 19, 20, 15, 16, 21)	

4. Other two-way designs obtained from balanced incomplete block designs. Balanced incomplete block designs in which the number of blocks (columns) is not an integral multiple of the number of treatments can be used to give designs with two accuracies for two-way elimination of heterogeneity. This is due to the fact that by suitable interchange of treatments in various columns it has been possible in every known case where $r \leq 10$ to express the design in a form such that in the matrix N, already referred to,

$$\Sigma_j n_{ij}^2 = \mu_2, \qquad i = 1, 2, \cdots, v, \tag{4.00}$$

$$\Sigma_j n_{ij} n_{uj} = \mu_e, \qquad i, u = 1, 2, \cdots, v; i \neq u, \tag{4.01}$$

where the treatments i and u are e-associates. These associates are similar to the associates defined by Bose and Nair [11]. Thus with respect to any treatment whatsoever, all the rest can be divided into two groups of associates with n_1 in the first group and n_2 in the second. If two treatments are e-associates, the number of treatments which are f-associates of the first and g-associates of the second is p_{fg}^e, independent of the particular pair of treatments started with. The relation of associates is reciprocal. The relations between the parameters can be derived, following Bose and Nair, as

$$\Sigma_e n_e = v - 1, \tag{4.1}$$

$$\Sigma_g p_{fg}^e = n_f \qquad \text{when } e \neq f, \tag{4.20}$$

$$= n_f - 1 \qquad \text{when } e = f, \tag{4.21}$$

$$n_e p_{fg}^e = n_f p_{ge}^f = n_g p_{ef}^g. \tag{4.3}$$

The normal equations for the estimation of treatment effects are (2.5), with

$$c_{ii} = r\left(1 - \frac{1}{k} + \frac{r}{bk}\right) - \frac{\mu_2}{b} = \alpha, \qquad i = 1, 2, \cdots, v, \tag{4.40}$$

$$c_{iu} = \frac{r}{v} - \frac{\lambda}{k} - \frac{\mu_e}{k} = \beta_e, \qquad i, u = 1, 2, \cdots, v; i \neq u, \tag{4.41}$$

where the treatments i and u are e-associates.

Following the method indicated by Bose [6], a solution of the normal equations is found to be

$$\alpha \hat{t}_i = Q_i - (\beta_1 A_{11} + \beta_2 A_{21}) Q_1(i) - (\beta_1 A_{12} + \beta_2 A_{22}) Q_2(i), \tag{4.5}$$

where $Q_e(i)$ denotes the sum of the Q's for all the e-associates of the treatment i, and (A_{ef}) is the inverse of the matrix (a_{ef}) whose elements are given by

$$a_{ef} = \alpha \delta_{ef} + \beta_e n_e + \beta_1 p_{e1}^f + \beta_2 p_{ef}^2, \qquad e, f = 1, 2, \tag{4.6}$$

where $\delta_{ef} = 1$ or 0, according as $e = f$ or $e \neq f$.

The analysis of variance can be obtained by substituting for $\hat{t}_i$ in Table I

$$V(\hat{t}_i - \hat{t}_u) = \frac{2\sigma^2}{\alpha}\{1 + \beta_1 A_{1e} + \beta_2 A_{2e}\} \tag{4.7}$$

if the treatments i and u are e-associates.

The designs considered here will be said to belong to the class Y_1. The parameters of some useful designs of this class are given in Table IIIa, and the actual designs in the Table IIIb.

The ratio of the variances of the two different kinds of comparisons is given by

$$R = \frac{1 + \beta_1 A_{11} + \beta_2 A_{21}}{1 + \beta_1 A_{12} + \beta_2 A_{22}}. \tag{4.8}$$

We shall now give a number of useful designs belonging to the class Y_1. One set of designs is obtained from the orthogonal series designs with the parameters

$$v = s^2, \quad b = s^2 + s, \quad r = s + 1, \quad k = s, \quad \lambda = 1, \tag{4.90}$$

the other parameters being

$$n_1 = s(s-1), \quad n_2 = s - 1, \quad \mu_1 = s + 2, \quad \mu_2 = s + 3, \tag{4.91}$$

$$(p^1_{fg}) = \begin{pmatrix} s(s-2) & s-1 \\ s-1 & 0 \end{pmatrix}, \quad (p^2_{fg}) = \begin{pmatrix} s(s-1) & 0 \\ 0 & s-2 \end{pmatrix}, \tag{4.92}$$

$$R = 1 + \frac{1}{s^3 + s^2 - s}. \tag{4.93}$$

These designs are obtained by using the difference sets of Bose [12]. He has shown that if $(d_1, d_2, \cdots, d_s)$ is the difference set corresponding to s, where s is a prime or power of a prime, then a solution of the balanced incomplete block design with parameters (4.90) is obtained as follows:

(i) $s^2 - 1$ blocks are obtained by developing the block $(d_1, d_2, \cdots, d_s)$ cyclically (mod $s^2 - 1$);

(ii) $s + 1$ other blocks are obtained from the block $(0, s + 1, 2(s + 1), \cdots, (s - 2)(s + 1), \infty)$ by adding successively the numbers $1, 2, \cdots, s + 1$, where ∞ remains invariant under the addition.

To convert this solution into a two-way design of the class Y_1, we keep the $s^2 - 1$ blocks (i) unchanged. Also the first two blocks of (ii) are kept unchanged, but in the others ∞ is successively moved to the left. Finally replace ∞ by s^2. For example, the difference set corresponding to $s = 3$ is (1, 6, 7), and hence the blocks of the design corresponding to $s = 3$ are (1, 6, 7), (2, 7, 8), (3, 8, 1), (4, 1, 2), (5, 2, 3), (6, 3, 4), (7, 4, 5), (8, 5, 6); (1, 5, 9), (2, 6, 9), (3, 9, 7), (9, 4, 8).

The method of identifying the associates is easy. Divide the treatments into s groups: $(1, 2, \cdots, s)$, $(s + 1, s + 2, \cdots, 2s)$, $\cdots$, $(s^2 - s + 1, s^2 - s + 2,$

$\cdots, s^2$). Any two treatments are 1-associates if they are in the different group and 2-associates if they are in the same group.

Bose's difference sets for $s = 2, 3, 4, 5, 7, 8$, and 9 are given below.

s	*Difference set*
2	1, 2
3	1, 6, 7
4	1, 3, 4, 12
5	1, 3, 16, 17, 20
7	1, 2, 5, 11, 31, 36, 38
8	1, 6, 8, 14, 38, 48, 49, 52
9	1, 13, 35, 48, 49, 66, 72, 74, 77

The parameters of some other designs of the class Y_1 are given in Table IIIa. The corresponding blocks are given in Table IIIb. In each case the treatments can be divided into a number of groups such that the treatments in different groups are 1-associates, and treatments in the same group are 2-associates. These groups are also shown in Table IIIb.

TABLE IIIa

Some designs of the class Y_1: Parameters

Reference no.	v, n_1,	b, n_2,	r, μ_1,	k, μ_2,	λ, R	(p^1_{fg})	(p^2_{fg})
1	10, 5,	15, 4,	6, 8,	4, 10,	2, 67/65	$\begin{pmatrix}0 & 4\\4 & 0\end{pmatrix}$	$\begin{pmatrix}5 & 0\\0 & 3\end{pmatrix}$
2	6, 4,	10, 1,	5, 8,	3, 9,	2, 39/38	$\begin{pmatrix}2 & 1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}4 & 0\\0 & 0\end{pmatrix}$
3	8, 6,	14, 1,	7, 12,	4, 13,	3, 83/82	$\begin{pmatrix}4 & 1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}6 & 0\\0 & 0\end{pmatrix}$
4	15, 10,	35, 4,	7, 16,	3, 17,	1, 171/170	$\begin{pmatrix}5 & 4\\4 & 0\end{pmatrix}$	$\begin{pmatrix}10 & 0\\0 & 3\end{pmatrix}$
5	10, 8,	18, 1,	9, 16,	5, 17,	4, 143/142	$\begin{pmatrix}6 & 1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}8 & 0\\0 & 0\end{pmatrix}$
6	16, 8,	24, 7,	9, 12,	6, 15,	3, 57/56	$\begin{pmatrix}0 & 7\\7 & 0\end{pmatrix}$	$\begin{pmatrix}8 & 0\\0 & 6\end{pmatrix}$

TABLE IIIb

Some designs of the class Y_1: Blocks and groups for identifying associates

Reference no.	Blocks	Groups
1	Develop the blocks $(1_1, 2_1, 4_2, 4_1)$, $(2_1, 1_2, 3_1, 4_2)$, $(1_2, 2_1, 2_2, 3_2)$, (mod 5), keeping the suffixes fixed	There are two groups. Treatments with the same suffixes belong to the same group
2	(6, 2, 3), (4, 3, 2), (3, 5, 4), (6, 5, 4), (5, 6, 2), (3, 1, 5), (2, 4, 1), (1, 6, 3), (4, 1, 6), (5, 2, 1)	There are three groups: (1, 2), (3, 4), (5, 6)
3	Develop the block (3, 5, 6, 7), (mod 7), and add the blocks (8, 2, 1, 4), (8, 5, 3, 2), (3, 8, 4, 6), (4, 8, 7, 5), (5, 6, 8, 1), (6, 7, 2, 8), (7, 1, 8, 3)	There are four groups: (1, 2), (3, 4), (5, 6), (7, 8)
4	(2, 1, 3), (4, 1, 5), (4, 6, 2), (8, 9, 1), (12, 8, 4), (8, 10, 2), (3, 13, 14), (5, 11, 14), (13, 6, 11), (14, 7, 9), (7, 11, 12), (7, 13, 10), (6, 7, 1), (2, 5, 7), (4, 7, 3), (1, 10, 11), (11, 3, 8), (2, 9, 11), (5, 8, 13), (1, 12, 13), (9, 4, 13), (12, 2, 14), (14, 6, 8), (10, 14, 4), (3, 5, 12), (15, 10, 5), (6, 15, 9), (15, 8, 7), (11, 4, 15), (13, 2, 15), (1, 14, 15), (3, 5, 6), (9, 3, 10), (5, 9, 12), (10, 12, 6)	There are three groups: (1, 2, 3, 4, 5), (6, 7, 8, 9, 10), (11, 12, 13, 14, 15)
5	(8, 2, 4, 10, 6), (7, 8, 10, 2, 1), (3, 8, 9, 4, 7), (9, 10, 1, 8, 5), (2, 5, 1, 10, 3), (10, 3, 4, 1, 6), (6, 1, 9, 5, 4), (5, 6, 8, 2, 9), (1, 6, 7, 3, 8), (4, 9, 2, 7, 10), (5, 10, 3, 9, 7), (6, 7, 2, 9, 1), (9, 1, 3, 4, 2), (4, 5, 8, 3, 2), (7, 2, 6, 5, 3), (3, 9, 10, 6, 8), (8, 7, 5, 1, 4), (10, 4, 7, 6, 5)	There are five groups: (1, 2), (3, 4), (5, 6), (7, 8), (9, 10)
6	(1, 2, 7, 8, 13, 14), (5, 13, 14, 12, 6, 11), (3, 10, 13, 9, 4, 14), (5, 6, 3, 15, 16, 4), (7, 9, 8, 10, 16, 15), (1, 2, 15, 16, 12, 11), (6, 3, 1, 15, 8, 13), (7, 15, 5, 13, 10, 12), (9, 11, 4, 13, 15, 2), (14, 7, 4, 16, 2, 5), (8, 14, 6, 9, 11, 16), (12, 3, 1, 10, 14, 16), (16, 4, 6, 1, 13, 7), (10, 8, 16, 11, 5, 13), (13, 16, 2, 12, 9, 3), (8, 5, 2, 14, 3, 15), (15, 12, 7, 6, 14, 9), (11, 4, 10, 14, 15, 1), (6, 5, 9, 2, 1, 10), (3, 1, 5, 7, 11, 9), (4, 1, 8, 5, 9, 12), (4, 7, 3, 11, 12, 8), (2, 8, 12, 4, 10, 6), (2, 6, 11, 3, 7, 10)	There are two groups: (1, 2, 3, 4, 5, 6, 7, 8), (9, 10, 11, 12, 13, 14, 15, 16)

5. Partial and extended partial Youden squares. We have seen how balanced incomplete block designs can be used for obtaining designs for two-way elimination of heterogeneity. In this and the following section we shall consider the use of partially balanced designs [11], [13] for the same purpose. The case when $b = v$ has already been considered by Bose and Kishen [14]. They call these designs partial Youden's squares. In this section we shall consider the case $b = mv$, $r = mk$, when $m \neq 1$; we may call these designs extended partial Youden's squares.

TABLE IV

Cyclic solutions to partially balanced incomplete block designs leading to partial and extended partial Youden's squares

Reference no.	v, n_1,	b, n_2,	r, λ_1,	k, λ_2	(p^1_{fg})	(p^2_{fg})	Solution
1	13, 6,	13, 6,	3, 1,	3, 0	$\begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$	$\begin{pmatrix} 3 & 3 \\ 3 & 2 \end{pmatrix}$	Develop (mod 13) the block (1, 3, 9).
2	15, 12,	30, 2,	6, 1,	3, 0	$\begin{pmatrix} 9 & 2 \\ 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 12 & 0 \\ 0 & 1 \end{pmatrix}$	Develop (mod 15) the blocks (1, 7, 9) and (1, 12, 15).
3	15, 12,	15, 2,	4, 1,	4, 0	$\begin{pmatrix} 9 & 2 \\ 2 & 0 \end{pmatrix}$	$\begin{pmatrix} 12 & 0 \\ 0 & 1 \end{pmatrix}$	Develop (mod 15) the block (1, 3, 4, 12).
4	17, 8,	17, 8,	8, 4,	8, 3	$\begin{pmatrix} 3 & 4 \\ 4 & 4 \end{pmatrix}$	$\begin{pmatrix} 4 & 4 \\ 4 & 3 \end{pmatrix}$	Develop (mod 17) the block (1, 9, 13, 15, 16, 8, 4, 2).
5	24, 20,	24, 3,	5, 1,	5, 0	$\begin{pmatrix} 16 & 3 \\ 3 & 0 \end{pmatrix}$	$\begin{pmatrix} 20 & 0 \\ 0 & 2 \end{pmatrix}$	Develop (mod 24) the block (1, 3, 16, 17, 20).
6	25, 12,	50, 12,	6, 1,	3, 0	$\begin{pmatrix} 5 & 6 \\ 6 & 6 \end{pmatrix}$	$\begin{pmatrix} 6 & 6 \\ 6 & 5 \end{pmatrix}$	Develop (mod 5, 5) the blocks (1, 5), (1, 4), (3, 1)] and [(3,5), (3,2), (4,3)].
7	26, 24,	26, 1,	9, 3,	9, 0	$\begin{pmatrix} 22 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 24 & 0 \\ 0 & 0 \end{pmatrix}$	Develop (mod 26) the block (0, 1, 2, 8, 11, 18, 20, 22, 23).
8	29, 14,	29, 14,	7, 2,	7, 1	$\begin{pmatrix} 6 & 7 \\ 7 & 7 \end{pmatrix}$	$\begin{pmatrix} 7 & 7 \\ 7 & 6 \end{pmatrix}$	Develop (mod 29) the block (1, 16, 24, 7, 25, 23, 20).
9	48, 42,	48, 5,	7, 1,	7, 0	$\begin{pmatrix} 36 & 5 \\ 5 & 0 \end{pmatrix}$	$\begin{pmatrix} 42 & 0 \\ 0 & 4 \end{pmatrix}$	Develop (mod 48) the block (1, 2, 5, 11, 31, 36, 38).
10	63, 56,	63. 6,	8, 1,	8, 0	$\begin{pmatrix} 49 & 6 \\ 6 & 0 \end{pmatrix}$	$\begin{pmatrix} 56 & 0 \\ 0 & 5 \end{pmatrix}$	Develop (mod 63) the block (1. 6, 8, 14, 38, 48, 49, 52).
11	80, 72,	80, 7,	9, 1,	9, 0	$\begin{pmatrix} 64 & 7 \\ 7 & 0 \end{pmatrix}$	$\begin{pmatrix} 72 & 0 \\ 0 & 6 \end{pmatrix}$	Develop (mod 80) the block (1, 13, 35, 48, 49, 66, 72, 74, 77).

Suppose there exists a partially balanced design with l different kinds of associates, and parameters $v, b, r, k; \lambda_1, \lambda_2, \cdots, \lambda_l; n_1, n_2, \cdots, n_l; p^e_{fg}$ $(e, f, g = 1, 2, \cdots, l)$. When $b = mv$, $r = mk$, then Smith and Hartley's process can be used just as in Section 3 to so modify the design that each treatment occurs just m times in each row (the columns constituting the blocks). In this case we have

$$(5.0) \qquad c_{ii} = r\left(1 + \frac{r}{bk}\right) - \frac{km^2}{b} - \frac{r}{k} = r\left(1 - \frac{1}{k}\right) = \alpha,$$

$$(5.1) \qquad c_{iu} = -\frac{km^2}{b} - \frac{\lambda_e}{k} + \frac{r^2}{bk} = -\frac{\lambda_e}{k} = \beta_e,$$

so that the normal equations take exactly the same form as for partially balanced incomplete block designs. Hence a solution of the normal equations is given by equations (4.5) and (4.6) of Section 4, with α and β_e now given by (5.0) and (5.1). The equation (4.7) is also valid. In case there are only two kinds of associates, the ratio of the variances of the two kinds of comparisons is given by (4.8).

When a cyclic solution to a partially balanced incomplete block design with $b = mv$, $r = mk$ is available, then it can be directly used as a two-way design without further modification. A number of cyclic solutions have been given by Bose and Nair [11]. Cyclic solutions to a number of new designs are given in Table IV. In each case $l = 2$.

6. Other two-way designs obtained from partially balanced incomplete block designs. Under certain conditions it is possible to use a partially balanced design with two types of associates to give a two-way design with two types of accuracies even when b is not an integral multiple of v. The necessary condition is

$$(6.0) \qquad \frac{v}{k} = \frac{b}{r} = n_1 + 1 \quad \text{or} \quad n_2 + 1.$$

In this case it has been found that in all cases of practical interest we can, by suitable interchanges within columns, arrange that

$$(6.1) \qquad \Sigma_j n_{ij}^2 = d, \qquad i = 1, 2, \cdots, v,$$

$$(6.2) \qquad \Sigma n_{ij} n_{uj} = \mu_e, \qquad i, u = 1, 2, \cdots, v; i \neq u; e = 1, 2,$$

where two treatments which are e-associates for the columns are also e-associates for the rows.

In this case

$$(6.3) \qquad c_{ii} = r\left(1 + \frac{r}{bk}\right) - \frac{d}{b} - \frac{r}{k} = \alpha, \qquad i = 1, 2, \cdots, v,$$

$$(6.4) \qquad c_{iu} = -\frac{\mu_e}{b} - \frac{\lambda_e}{k} + \frac{r^2}{bk} = \beta_e, \qquad i, u = 1, 2, \cdots, v; i \neq u.$$

The analysis is the same as in Section 4, the equations (4.5), (4.6), (4.7), (4.8) remaining valid but α and β now given by (6.3) and (6.4). The analysis of variance is obtained by substituting for $\hat{t}_i$ in Table I.

The designs considered here may be said to belong to class Y_2. Some designs of this class are given below in Tables Va and Vb. The parameters are given in Table Va whereas the actual solutions appear in Table Vb. In this case the representation is such that two treatments in the same group are 1-associates, whereas two treatments in different groups are 2-associates. These groups are also shown in Table Vb.

TABLE Va

Some designs of the class Y_2: Parameters

Reference no.	v, λ_1,	b, λ_2,	r, μ_1,	k, μ_2,	n_1, d	n_2	(p^1_{jg})	(p^2_{jg})
1	12, 5,	10, 2,	5, 5,	6, 4,	1, 5	10	$\begin{pmatrix} 0 & 0 \\ 0 & 10 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 8 \end{pmatrix}$
2	15, 0,	25, 1,	5, 11,	3, 7,	4, 11	10	$\begin{pmatrix} 3 & 0 \\ 0 & 10 \end{pmatrix}$	$\begin{pmatrix} 0 & 4 \\ 4 & 5 \end{pmatrix}$

TABLE Vb

Some designs of the class Y_2: Blocks and groups for identifying associates

Reference no.	Blocks	Groups
1	(8, 11, 5, 7, 1, 2), (2, 3, 1, 8, 7, 9), (3, 4, 10, 9, 2, 8), (4, 10, 11, 5, 9, 3), (5, 1, 4, 11, 10, 7), (6, 5, 8, 2, 12, 11), (9, 12, 7, 6, 3, 1), (10, 6, 2, 12, 8, 4), (11, 9, 12, 3, 6, 5), (12, 7, 6, 1, 4, 10)	There are six groups: (1, 7), (2, 8), (4, 9), (9, 10), (5, 11), (6, 12).
2	(10, 6, 4), (3, 7, 5), (11, 2, 13), (1, 9, 8), (14, 12, 15), (11, 12, 5), (3, 6, 8), (14, 7, 4), (10, 9, 13), (1, 2, 15), (1, 7, 13), (3, 2, 4), (10, 12, 8), (14, 9, 5), (11, 6, 15), (15, 3, 9), (4, 1, 12), (5, 10, 2), (8, 11, 7), (13, 14, 6), (2, 8, 14), (6, 5, 1), (9, 4, 11), (7, 5, 10), (12, 13, 3)	There are three groups: (1, 3, 10, 11, 14), (2, 6, 7, 9, 12), (4, 5, 8, 13, 15).

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