ON THE DURATION OF RANDOM WALKS[1]

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Summary and introduction. In a recent paper [1] the author investigated the mean number of steps in random walks in n-dimensional domains. The purpose of the present article is to generalize those results by applying similar methods to the study of the moment generating function for the number of steps and of its distribution function. As an application explicit asymptotic expressions for the variance in special cases and estimates for the likelihood of very long walks are obtained.

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The walks take place in an open bounded domain B of n-dimensional Euclidean space E with boundary C. A point moves in E according to a given transition probability law $F(y, x)$. Here x and y are points of E with coordinates x_i, $i = 1, 2, \cdots, n$, and y_i, $i = 1, 2, \cdots, n$, and $F(y, x)$ is the probability that a jump known to start at x end at a point all of whose coordinates are less than the corresponding ones of y. The function $F(y, x)$ is a distribution function with respect to y, and it is assumed to be Borel measurable with respect to all variables. Let $N = N_x$ be the number of steps in a random walk that begins at a point x of B and ends with the step on which the moving point leaves B for the first time. If the probability of the moving point eventually leaving B is equal to one, then N is a random variable. It is called the *duration* of the walk. It is useful to extend the definition of N by setting

$$N_x = 0, \qquad x \,\varepsilon\, E - B.$$

1. The moment generating function of the duration.

The probability distribution of the duration, given by the functions

$$p_k(x) = Pr\{N_x = k\}, \qquad k = 0, 1, 2, \cdots, \tag{1.1}$$

satisfies the recursion relations

$$\begin{aligned} p_{k+1}(x) &= \begin{cases} \int_B p_k(y)\, dF(y, x), & x \,\varepsilon\, B, \\ 0, & x \,\varepsilon\, E - B, \end{cases} \\ p_0(x) &= \begin{cases} 0, & x \,\varepsilon\, B, \\ 1, & x \,\varepsilon\, E - B. \end{cases} \end{aligned} \tag{1.2}$$

Here and in the sequel all Stieltjes differentials are formed with respect to the first argument.

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We need some hypothesis sufficient to ensure the relation $\sum_{k=0}^{\infty} p_k(x) = 1$ and the existence of the moment generating function of N_x. This is the purpose of

ASSUMPTION A. *There exists a positive integer m and a positive number* $c < 1$, *both independent of x, such that*

$$Pr\{N_x \geq m\} \leq c$$

for all x in B.

From this assumption, which is slightly more general than the corresponding condition in [1], the equality $\sum_{k=0}^{\infty} p_k(x) = 1$ follows by a simple argument similar to that in [2], pp. 431–432. In fact, the last inequality implies

$$Pr\{N_x \geq jm\} \leq c^j$$

and therefore

$$\lim_{j\to\infty} \sum_{k=0}^{mj} p_k(x) = \lim_{j\to\infty} [1 - Pr\{N_x \geq jm\}] \geq \lim_{j\to\infty} (1 - c^j) = 1.$$

The aim of this section is the following theorem:

THEOREM 1. *If Assumption A is satisfied, then the moment generating function* $\phi(s, x) = \sum_{j=0}^{\infty} e^{sj} p_j(x)$ *of the duration* N_x *exists in a complex neighborhood of* $s = 0$ *and is the unique solution of the integral equation problem*

$$\phi(s, x) = \begin{cases} e^s \int_B \phi(s, y)\, dF(y, x), & x \in B, \\ 1, & x \in E - B. \end{cases} \tag{1.3}$$

Because of later need we prove a slightly more general result.

LEMMA 1. *If Assumption A is satisfied and f(x) is a real function such that* $|f(x)| \leq K$ *in* $E - B$, *then the integral equation problem*

$$u(s, x) = \begin{cases} e^s \int_B u(s, y)\, dF(y, x), & x \in B, \\ f(x), & x \in E - B, \end{cases} \tag{1.4}$$

possesses for $Re\ s < s_0 (s_0 > 0)$ *a unique solution. This solution satisfies the inequality*

$$|u(s, x)| \leq \phi(Re\ s, x) \cdot K, \qquad x \in B, \tag{1.5}$$

where $\phi(s, x)$, *the moment generating function of the duration, is the solution of* (1.3).

PROOF. Assume, at first, that $f(x)$ is nonnegative, and that s is real. Set

$$u_0(x) = \begin{cases} 0, & x \in B, \\ f(x), & x \in E - B, \end{cases}$$

$$u_{k+1}(x) = \begin{cases} e^s \int_B u_k(y)\, dF(y, x), & x \in B, \\ f(x), & x \in E - B. \end{cases} \tag{1.6}$$

Then

$$(1.7) \qquad u_k(x) = \sum_{j=1}^{k} e^{js} q_j(x), \qquad x \varepsilon B, \qquad k = 1, 2, \cdots,$$

where, for x in B,

$$(1.8) \qquad q_1(x) = \int_{E-B} f(y)\, dF(y, x), \; q_{j+1}(x) = \int_B q_j(y)\, dF(y, x).$$

The $q_j(x)$ are nonnegative, and the $u_k(x)$ form therefore a nondecreasing sequence. Iterating (1.6) m times we find, for $k > m$,

$$(1.9) \qquad u_{k+1}(x) = e^{ms} \int_B u_{k-m}(x)\, dF_m(y, x) + \chi_m(x), \qquad x \varepsilon B,$$

where

$$F_1(y, x) = F(y, x), \qquad F_m(y, x) = \int_B F_{m-1}(y, z)\, dF(z, x),$$

and the $\chi_m(x)$ are bounded and nonnegative. If x is in B, then, by Assumption A,

$$\int_B dF_m(y, x) = Pr\{N_x > m\} \leq c < 1.$$

Let

$$L_k = \underset{x \varepsilon B}{\text{l. u. b.}}\, u_k(x), \qquad M = \underset{x \varepsilon B}{\text{l. u. b.}}\, \chi_m(x).$$

(L_k and M depend on s.) Then, by (1.9),

$$L_{k+1} \leq e^{ms} L_k c + M.$$

Hence,

$$L_{k+1} \leq M/(1 - e^{ms} c),$$

and, therefore, the nondecreasing sequence $u_k(x)$ is bounded for all s for which $e^{ms}c < 1$. Thus, it tends to a limit $u(x) = u(s, x)$, which satisfies (1.4) and can be written, by (1.7), in the form

$$(1.10) \qquad u(s, x) = \sum_{j=1}^{\infty} e^{js}\, q_j(x).$$

Since this is a power series in e^s it converges for complex s also, as long as $Re\ s < s_0 = -\log c/m$. Furthermore, we see by comparison of (1.8) and (1.2) that for $f(x) \equiv 1$ we have $q_k(x) = p_k(x)$ and $u(s, x) = \phi(s, x)$.

To prove the uniqueness, it suffices to show that $f(x) \equiv 0$ implies $u(x) \equiv 0$. By iteration of (1.4) we find, for $f(x) \equiv 0$,

$$u(x) = e^{ms} \int_B u(y)\, dF_m(y, x),$$

and hence

$$\underset{x \varepsilon B}{\text{l. u. b.}} \mid u(x) \mid \leq \mid e^{ms} \mid c \cdot \underset{y \varepsilon B}{\text{l. u. b.}} \mid u(y) \mid .$$

For values of s such that $\mid e^{ms} \mid c < 1$ this implies, indeed, that $u(x) \equiv 0$.

If $f(x)$ is not necessarily positive, then we consider the integral equation problems

$$u^{(1)} = \begin{cases} e^s \int_B u^{(1)} \, dF & \text{in } B, \\ K & \text{in } E - B, \end{cases}$$

$$u^{(2)} = \begin{cases} e^s \int_B u^{(2)} \, dF & \text{in } B, \\ K - f(x) & \text{in } E - B, \end{cases}$$

which do have unique solutions by what has been proved already. $u = u^{(1)} - u^{(2)}$ is therefore the unique solution of the original problem. We note that this argument also extends the validity of the formulas (1.8) and (1.10) to the case that $f(x)$ is not necessarily positive.

Finally, the inequality (1.5) follows easily from (1.8), (1.2), and (1.10), since

$$\mid u(x) \mid \leqq \sum_{j=1}^{\infty} e^{jRe\, s} \mid q_j(x) \mid \leqq K \sum_{j=1}^{\infty} e^{jRe\, s} \, p_j(x) = K\phi(Re\ s, x).$$

This completes the proof of Lemma 1. Theorem 1 implies that all moments $M_k(x)$ of N_x exist.

THEOREM 2. *The* kth *moment* $M_k(x)$ *of the duration* N_x *satisfies, for* $k > 0$, *the integral equation problem*

$$M_k(x) = \begin{cases} \int_B M_k(y) \, dF(y, x) + f_k(x), & x \varepsilon B, \\ 0, & x \varepsilon E - B, \end{cases} \tag{1.11}$$

where

$$f_k(x) = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} M_{k-j}(x). \tag{1.12}$$

PROOF. From the definition

$$M_k(x) = \sum_{j=0}^{\infty} j^k \, p_j(x)$$

follows by an application of (1.2) that, for x in B,

$$\int_E M_k(y) \, dF(y, x) = \sum_{j=1}^{\infty} (j - 1)^k \, p_j(x). \tag{1.13}$$

Expansion of the binomial expression in the second member followed by an interchange of summations proves the theorem.

These integral equations for the moments form an inductive sequence, since $f_k(x)$ depends only on the moments of order $j < k$. The equation for $M_1(x)$ was the main subject of [1].

2. An asymptotic differential equation for the moment generating function. In the important case that the transition probability $F(y, x)$ is strongly concentrated about x, the integral equation (1.3) will now be shown to be approximately equivalent to a differential equation. To do this it will be assumed, as in [1] and [2], that $F(y, x) = F(y, x, \mu)$ depends on a small positive parameter μ in such a way that the following three hypotheses are satisfied.

Assumption B. *Denote by $a_i(x, \mu)$, $b_{ik}(x, \mu)$, $i, k = 1, 2, \cdots, n$, the first and second moments of $F(y, x, \mu)$ about x. Then*

$$a_i(x, \mu) = \alpha_i(x)\mu + o(\mu), \tag{2.1}$$

$$b_{ik}(x, \mu) = \beta_{ik}(x)\mu + o(\mu). \tag{2.2}$$

These relations hold uniformly for x in B. The $\alpha_i(x)$ and $\beta_{ik}(x)$ are twice continuously differentiable in $B + C$.

Assumption L. *Let $K_r(x)$ denote the sphere of radius r with center at x. Then*

$$\int_{B-K_r(x)} (y_i - x_i)(y_k - x_k)\, dF(y, x, \mu) = o(\mu), \qquad i, k = 1, 2, \cdots, n,$$

uniformly with respect to x in B, for any fixed $r > 0$.

These assumptions could very likely be weakened to the equivalent of the analogous hypotheses in [12].

Assumption E. *The matrix $\{\beta_{ik}(x)\}$, which is obviously nonnegative, is positive definite in $B + C$.*

For what follows we also require a certain degree of smoothness of the boundary.

Assumption S. *The boundary C has a continuously turning tangent plane.* (This restriction could be considerably weakened; e.g., by inserting the word "piecewise", cf. [2], p. 438.)

We prove first

Lemma 2. *Assumptions B, L, and E imply Assumption A, at least for sufficiently small μ.*

Proof. To simplify the notation we give the proof first for the one-dimensional case, in which we can write α, β, x, y for $\alpha_i, \beta_{ik}, x_i, y_i$. Using Assumptions B and L we have, for any $\epsilon > 0$,

$$\beta\mu = \int_{-\infty}^{\infty} (y - x)^2\, dF(y, x, \mu) + o(\mu) = \int_x^{x+\epsilon} (y - x)^2\, dF$$

$$+ \int_{x-\epsilon}^{x} (y - x)^2\, dF + o(\mu) \le \epsilon \int_x^{x+\epsilon} (y - x)\, dF - \epsilon\int_{x-\epsilon}^{x} (y - x)\, dF + o(\mu)$$

and also

$$\alpha\mu = \int_x^{x+\epsilon} (y - x)\, dF + \int_{x-\epsilon}^{x} (y - x)\, dF + o(\mu).$$

Multiplying the last equality by ϵ and adding it to the preceding inequality, we find

$$(\beta - \epsilon\alpha)\mu + o(\mu) \leq 2\epsilon \int_x^{x+\epsilon} (y - x)\, dF.$$

Since $\beta(x) \geq$ const. > 0 in B, by assumption E, we see, by first choosing ϵ sufficiently small, that for small μ

$$\int_x^{x+\epsilon} (y - x)\, dF \geq C \cdot \mu, \qquad x \in B,$$

where C is an (arbitrary) positive constant.

On the other hand, for $\mu^2 < \epsilon$,

$$\int_x^{x+\epsilon} (y - x)\, dF = \int_x^{x+\mu^2} (y - x)\, dF + \int_{x+\mu^2}^{x+\epsilon} (y - x)\, dF \leq \mu^2 + \epsilon \int_{x+\mu^2}^{\infty} dF.$$

Combining this with the preceding inequality, we have

$$Pr\{y - x \geq \mu^2\} \geq C_1\mu, \qquad x \in B,$$

where C_1 is another constant. Hence Assumption A is satisfied, if m is chosen greater than the diameter D of B divided by μ^2, and $c = 1 - (C_1\mu)^{D/\mu^2}$.

In more than one dimension the same argument can be applied in any one of the coordinate directions. The changes necessary concern only the notation. Thus the lemma is proved.

Denote by $\psi(s, x, \mu)$ the moment generating function of the random variable

$$t = t_x = \mu N_x .$$

Obviously

$$\psi(s, x, \mu) = \phi(\mu s, x, \mu). \tag{2.3}$$

Let furthermore $L[u]$ be an abbreviation for the operator

$$L[u] = \tfrac{1}{2} \sum_{i,k=1}^{n} \beta_{ik}(x) \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{j=1}^{n} \alpha_j(x) \frac{\partial u}{\partial x_j}. \tag{2.4}$$

Then the following theorem will be proved.

THEOREM 3. *If Assumptions B, L, E, and S are satisfied, then the moment generating function $\psi(s, x, \mu)$ of the random variable $t = \mu N$ satisfies the limit relation*

$$\lim_{\mu \to 0} \psi(s, x, \mu) = u(s, x) \tag{2.5}$$

uniformly for x in B, $|s| \leq s_1$, where $u(s, x)$ is the solution of the problem

$$L[u] + su = 0, \quad x \varepsilon B, \qquad u = 1, \quad x \varepsilon E - B. \tag{2.6}$$

Before we can prove the limit relation (2.5) we have to prove separately the weaker statement that $\psi(s, x, \mu)$ remains bounded as $\mu \rightarrow 0$.

LEMMA 3. *If Assumptions B, L, E, and S are satisfied, then there exist two positive constants C and C', independent of μ, such that*

$$| \psi(s, x, \mu) | \leq C \quad \text{for} \quad |s| \leq C'. \tag{2.7}$$

PROOF. From the results of [1], in particular from Lemma 3, Theorem 1, and Theorem 2 of that paper, it follows that the solution of the problem

$$M(x) = \begin{cases} \int_B M(y)\, dF(y, x, \mu) + f(x), & x \varepsilon B, \\ 0, & x \varepsilon E - B, \end{cases} \tag{2.8}$$

satisfies the inequality

$$| M(x) | \leq C_1 \operatorname*{l.u.b}_{x \varepsilon B} | f(x)|/\mu, \tag{2.9}$$

where C_1 is a constant independent of μ. This remains true if $f(x)$ depends boundedly on μ. We wish to apply this inequality to the integral equations (1.11). To that end we first note that

$$| f_k(x) | \leq kM_{k-1}(x), \tag{2.10}$$

for $f_k(x)$ can be written, by (1.11) and (1.13), in the form

$$f_k(x) = M_k(x) - \sum_{j=1}^{\infty} (j - 1)^k p_j(x) = \sum_{j=1}^{\infty} [j^k - (j - 1)^k] p_j(x)$$

$$= k \sum_{j=1}^{\infty} j^{*k-1} p_j(x),$$

where $j - 1 < j^* < j$, and, therefore,

$$0 \leq f_k(x) \leq k \sum_{j=1}^{\infty} j^{k-1} p_j(x) = kM_{k-1}(x).$$

Applying (2.9) and (2.10) inductively to (1.11) we obtain

$$| M_k(x) | \leq k!(C_1/\mu)^k. \tag{2.11}$$

Substitution of this inequality into the formula

$$\psi(s, x, \mu) = \sum_{k=0}^{\infty} \frac{M_k}{k!} (\mu s)^k$$

yields

$$| \psi (s, x, \mu) | \leq \sum_{k=0}^{\infty} (C_1 s)^k = \frac{1}{1 - C_1 s}$$

for $| s | < C_1$. This proves Lemma 3.

PROOF OF THEOREM 3. The basic idea of the proof is similar to that of Theorem 2 in [1] and thus to Petrowsky's reasoning in [2]; we show that $u(x)$ satisfies an integral equation little different from that for $\psi(s, x, \mu)$, and conclude from that fact that the two functions are nearly the same.

We first replace $u(x)$ by a slightly different function $u_\delta(x)$ defined in a larger domain B'', in order to avoid extraneous difficulties near the boundary C. This can be done by constructing a twice continuously differentiable mapping

$$x_i' = f(x, \delta), \qquad x \varepsilon B, \qquad i = 1, 2, \cdots, n,$$

which is continuous in δ, for $\delta \geq 0$, together with its first and second derivatives with respect to x, and has the following properties.

(a) It reduces to the identity for $\delta \to 0$.

(b) It is, for all δ, the identity transformation in a subdomain B' of B that tends to B as $\delta \to 0$.

(c) It maps B onto a domain B'' containing B in its interior for $\delta > 0$.

For the explicit construction of such a mapping with the help of Assumption S we refer to [2].

If we define $u_\delta(x)$ in B by

$$u_\delta(x') = u(x), \qquad x \varepsilon B,$$

then this new function is defined and twice continuously differentiable in B''. It tends to $u(x)$, uniformly in B, together with its first and second derivatives. We extend the definition into the whole space E by setting

$$u_\delta(x) = 1, \qquad x \varepsilon E - B.$$

Next, it can be shown that, for any $\epsilon > 0$, we can choose first a $\delta > 0$ and then a $\mu_0 > 0$ such that, for $| s | \leq s_0$,

$$\int_E u_\delta(y)\, dF(y, x) = (1 - \mu s) u_\delta(x) + \mu g(s, x, \mu), \qquad x \varepsilon B, \tag{2.12}$$

where

$$| g(s, x, \mu) | < \epsilon, \tag{2.13}$$

provided $\mu \leq \mu_0$. The proof of this statement resembles so much the analogous arguments in [1] and [2] that it will be omitted here. (Formula (2.12) is essentially the result of expanding $u_\delta(x)$ about x by Taylor's formula up to quadratic terms and applying Assumptions B and L.)

Because of the definition of $u_\delta(x)$ it can also be assumed that δ has been chosen so small that

(2.14) $$| u_\delta - u | \leq \epsilon, \qquad x \in B,$$

(2.15) $$| u_\delta - 1 | \leq \epsilon, \qquad x \in E - B.$$

For μ_0 sufficiently small,

$$| (1 - \mu s - e^{-\mu s}) u_\delta | \leq \mu\epsilon, \qquad x \in B, \qquad \mu \leq \mu_0 .$$

Therefore we can write, instead of (2.12),

$$e^{\mu s} \int_E u_\delta(y)\, dF(y, x) = u_\delta(x) + \mu h(s, x, \mu), \qquad x \in B,$$

with

(2.16) $$| h(s, x, \mu) | \leq 3\epsilon.$$

We now split u_δ into the sum $u_\delta = u_\delta^{(1)} + u_\delta^{(2)}$, where

(2.17) $$u_\delta^{(1)} = e^{\mu s} \int_E u_\delta^{(1)}\, dF \quad \text{in} \quad B, \qquad u_\delta^{(1)} = u_\delta \quad \text{in} \quad E - B,$$

(2.18) $$u_\delta^{(2)} = e^{\mu s} \int_E u_\delta^{(2)}\, dF + \mu h \quad \text{in} \quad B, \qquad u_\delta^{(2)} = 0 \quad \text{in} \quad E - B.$$

To estimate $u_\delta^{(1)}$ we subtract it from (1.3) with $s\mu$ substituted for s, and use (2.15), (2.3), Lemma 1, and (2.7). This yields

(2.19) $$| u_\delta^{(1)} - \psi | \leq CK\epsilon, \qquad x \in B.$$

This implies, in particular, that $u_\delta^{(2)}$ is bounded as $\mu \to 0$. Therefore (2.18) can be written, for sufficiently small μ_0, in the form

$$u_\delta^{(2)} = \int_E u_\delta^{(2)}\, dF + \mu(s u_\delta^{(2)} - h^*) \quad \text{in} \quad B, \qquad u_\delta^{(2)} = 0 \quad \text{in} \quad E - B,$$

where

$$| h^* | = | h^*(s, x, \mu) | \leq 4\epsilon.$$

Application of (2.9) yields

$$\text{l.u.b.}\, | u_\delta^{(2)} | \leq C_1(| s |\, \text{l.u.b.}\, | u_\delta^{(2)} | + 4\epsilon),$$

i.e.,

(2.20) $$| u_\delta^{(2)} | \leq 4\epsilon C_1/(1 - C_1 | s |)$$

for $| s | \leq s_0 < 1/C_1$. Combining (2.19) and (2.20) we obtain the inequality

$$| u - \psi | \leq \text{const.}\ \epsilon, \qquad x \in B,$$

which proves Theorem 3.

THEOREM 4. *If Assumptions B, L, E, and S are satisfied, then the kth moment $M_k(x, \mu)$ of the duration N_x satisfies, uniformly in B, the relation*

$$\lim_{\mu \to 0} \mu^k M_k(x, \mu) = m_k(x), \qquad k \geq 1,$$

where the $m_k(x)$ are defined recursively by the conditions

$$m_0(x) = 1,$$

and, for $k \geq 1$,

$$L[m_k] + km_{k-1} = 0, \qquad x \varepsilon B,$$

$$m_k = 0, \qquad x \text{ on } C.$$

PROOF. The solution $u(x, s)$ of (2.6) is connected with the functions $m_j(x)$ by the relation

$$u(s, x) = \sum_{j=0}^{\infty} \frac{m_j(x)}{j!} s^j,$$

as can be seen by replacing $u(x, s)$ in (2.6) by its series in powers of s and collect ing coefficients of like powers. By Theorem 3 the function

$$\psi(s, x, \mu) = \sum_{j=0}^{\infty} \frac{\mu^j M_j(x)}{j!} s^j$$

tends uniformly to $u(x, s)$ as $\mu \to 0$; hence the coefficients of the first power series are the uniform limits of those of the second. This proves Theorem 4.

For $j = 1$, Theorem 4 was proved in [1].

3. An asymptotic differential equation for the distribution function of the duration. Let

$$P(t, x, \mu) = Pr\{\mu N_x \leq t\}$$

be the distribution function of the random variable μN. From Theorem 3 we conclude by means of the continuity theorem for moment generating functions (see [3]) that there exists a distribution function $Q(t, x)$ of t such that

$$\lim_{\mu \to 0} P(t, x, \mu) = Q(t, x) \tag{3.1}$$

at all continuity points of $Q(t, x)$ with respect to t, and that $u(s, x)$ is the moment generating function of $Q(t, x)$.

The probability

$$P(t, x, \mu) = \sum_{k \leq t/\mu} p_k(x) \tag{3.2}$$

satisfies, because of (1.2), the recursive relations

$$P(t + \mu, x, \mu) = \int_E P(t, y, \mu)\, dF(y, x, \mu), \qquad x \varepsilon B, \qquad t = 0, \mu, 2\mu, \cdots,$$

$$P(0, x, \mu) = 0, \qquad x \varepsilon B, \tag{3.3}$$

$$P(t, x, \mu) = 1, \qquad x \varepsilon E - B, \qquad t > 0.$$

From these and (3.1) it is easy to obtain, *in a purely formal way*, the differential equation (3.5) for $Q(t, x)$. The same result can be made plausible by setting

$$u(s, x) = \int_0^\infty e^{st} \, dQ(t, x) \tag{3.4}$$

in (2.6) and operating *formally* on the Stieltjes differential. Our aim in this section is to give a proof of (3.5). In spite of the plausibility of the result the proof is somewhat long, because the problem combines the features of what Khintchine calls, in [4], the first and second diffusion problems.

A feasible approach to our problem, different from that of this paper, could be based on the remark that $u(s, x)$ and therefore $Q(t, x)$ depend only on the functions $\alpha_i(x)$, $\beta_{ik}(x)$, so that $F(y, x, \mu)$ can be chosen in many different ways as far as the properties of $Q(t, x)$ are concerned. (This is an instance of the "invariance principle", used systematically, in a different context, by Kac and Erdös, cf. [5].) The most natural choice for $F(y, x, \mu)$ is the one obtained from the continuous Markov process associated with the differential equation (3.5) by considering that process at discrete time values $t = 0, \mu, 2\mu, \cdots$, only. This approach has not been chosen here, first, because we wish to preserve uniformity of method, and secondly, because the theory of such Markov processes does not seem to have been established in sufficient completeness for the n-dimensional case. (In one dimension, the proof that such a continuous process exists was given by Feller in [12]. This has been partially generalized to n dimensions by Dressel [6]. The proof that the duration of this continuous process satisfies the differential equation of (3.5) in the one-dimensional case is contained in the article [13] by Fortet.)

THEOREM 5. *If the assumptions B, L, E, and S are satisfied, then*

$$\lim_{\mu \to 0} Pr\{\mu N_x \leq t\} = Q(t, x)$$

exists and is the solution of the differential equation problem

$$\begin{aligned} L[Q] - \frac{\partial Q}{\partial t} &= 0, && t > 0, && x \, \varepsilon \, B, \\ Q &= 1, && t > 0, && x \text{ on } C, \\ Q &= 0, && t = 0, && x \, \varepsilon \, B. \end{aligned} \tag{3.5}$$

The convergence is uniform in $B + C$, for any interval $0 < t_0 \leq t \leq t_1$.

The proof of this theorem is based on two lemmas.

LEMMA 4. *The distribution function $Q(t, x)$ is a continuous function of all arguments combined, for $t > 0$, $x \, \varepsilon \, B + C$, and for $t = 0$, $x \, \varepsilon \, B$.*

PROOF. Let

$$g(t, x, T) = \frac{1}{2T} \int_{-T}^{T} u(is, x) e^{-ist} \, ds. \tag{3.6}$$

It is known that this function is real and tends to a limit as $T \to \infty$. For fixed x the distribution $Q(t, x)$ is continuous in t if and only if

$$\lim_{T \to \infty} g(t, x, T) = 0. \tag{3.7}$$

(See, e.g., [7], p. 24, for these statements.) Also,

$$| g(t, x, T)| \leq 1, \tag{3.8}$$

since $u(is, x)$, as a characteristic function, is numerically less than one. By (2.6) the function $g(t, x, T)$ satisfies for all T the differential equation

$$L[g] - \frac{\partial g}{\partial t} = 0, \qquad x \varepsilon B, \qquad -\infty < t < \infty. \tag{3.9}$$

From (3.8) and (3.9) we can conclude (cf. [8], p. 383–384) that $\partial g/\partial t$ is uniformly bounded for all T in any finite t-interval and for x in any closed subdomain of B. Therefore the limit of g, as $T \to \infty$, is a continuous function of t. On the other hand, since $Q(t, x)$ as a distribution function has at most a denumerable set of discontinuities, $\lim_{T\to\infty} g$ is zero for fixed x, except possibly at a denumerable set of t-values. Being continuous, the limit of g must therefore be zero *everywhere* in the domain considered, i.e., $Q(t, x)$ is for all x in B and for all t a continuous function of t. (The result of Gevrey [8], referred to above, is proved in that paper only for differential equations whose second order terms form Laplace's operator. A generalization sufficient for our needs can be established by combining Gevrey's arguments with the results of [6].) In $E - B$ the distribution functions $P(t, x, \mu)$—and therefore $Q(t, x)$, their limit as $\mu \to 0$— are identically 1 for $t > 0$. Hence $Q(t, x)$ is, for $t > 0$, a continuous function of t in the closed domain $B + C$.

To prove that $Q(t, x)$ is continuous in x also, it suffices to remember that $u(is, x)$, its characteristic function, is a continuous function of x at $s = 0$. By the continuity theorem for characteristic functions ([7], p. 30) the corresponding distribution function $Q(t, x)$ is therefore continuous in x for all $t > 0$. The continuity is uniform with respect to x in every continuity interval of t ([7], p. 31) and therefore $Q(t, x)$ is continuous in t and x combined for $t > 0$ and x in $B + C$, as well as for $t = 0$, $x \varepsilon B$. This proves the lemma.

COROLLARY. *The convergence of $P(t, x, \mu)$ to $Q(t, x)$, as $\mu \to 0$, is uniform for x in $B + C$ and $0 < t_0 \leq t \leq t_1$.* For, by a similar argument to that used in the preceding paragraph, it is seen that $P(t, x, \mu)$ is a continuous function of all arguments combined at $\mu = 0$, and this implies *uniform* continuity in the designated domain. This proves the last sentence of Theorem 5.

LEMMA 5. *Let $u_k(x, \mu)$ satisfy the recursive relations*

$$\begin{aligned} u_{k+1}(x, \mu) &= \int_E u_k(y, \mu)\, dF(y, x, \mu) + \mu a_k(x, \mu), & x \varepsilon B, \\ u_k(x, \mu) &= b_k(x, \mu), & x \varepsilon E - B, \end{aligned} \tag{3.10}$$

and let $|a_k|$, $|b_k|$, *and* $|u_0|$ *be less than a constant* M. *Then, if Assumptions B, L, E, and S are satisfied, the inequality*

$$|u_k(x, \mu)| \leq C \cdot M$$

holds, where C *is a constant independent of* M.

Proof. Assume, at first, that b_k and u_0 are identically zero, and that $a_k \equiv 1/\mu$. We denote the solution for this special case by u_k^*. It was proved in [1], Lemma 2, that $u_k^*(x)$ tends monotonically to the first moment $M_1(x)$ of N_x, as $k \to \infty$. From Theorem 4 we know that $\lim_{\mu \to 0} \mu M_1(x) = m_1(x)$, uniformly in B. Hence, $0 < u_k^* \leq C/\mu$.

Next, we drop the assumption $a_k \equiv 1/\mu$ and call the solution of the integral relation (3.10), in that case, $u_k^{(1)}$. Then the function $u_k^{**} = \mu M u_k^* - u_k^{(1)}$ solves the problem

$$u_{k+1}^{**} = \int_B u_k^{**}\, dF + \mu(M - a_k) \quad \text{in} \quad B,$$

$$u_k^{**} = 0 \quad \text{in} \quad E - B; \; u_0^* = 0 \quad \text{in} \quad B.$$

Since $M - a_k \geq 0$, it follows that $u_k^{**} \geq 0$ in B for all k, i.e., $u_1^{(1)} \leq$ const. M. The inequality $-u_1^{(1)} \leq$ const. M is proved analogously. Thus the lemma is proved in this special case.

Now we take the solution $u_k^{(2)}(x, \mu)$ of the special case that $a_k \equiv u_0 \equiv 0$. Here we obtain immediately by recursion the inequality $|u_k^{(2)}(x, \mu)| \leq M$. The solution $u_k^{(3)}(x, \mu)$ of the special case $a_k \equiv b_k \equiv 0$ also satisfies trivially the inequality $|u_k^{(3)}(x, \mu)| \leq M$.

Since the solution in the general case is the sum of three solutions corresponding to the three special cases, the lemma is proved.

Proof of Theorem 5. Instead of comparing $P(t, x, \mu)$ directly with the solution of (3.5) we introduce the solution v of the problem

$$L[v] - \frac{\partial v}{\partial t} = 0 \quad \text{in} \quad B, \qquad t > 0, \qquad x \in B, \tag{3.11}$$

$$v(0, x) = Q(t_0, x), \qquad x \in B, \tag{3.12}$$

$$v(t, x) = 1, \qquad x \in E - B. \tag{3.13}$$

By this device we avoid difficulties connected with the discontinuity in the boundary conditions in (3.5) at $t = 0$, $x \in C$.

As in the proof of Theorem 3 we replace $v(t, x)$ by the function $v_\delta(t, x)$ defined by

$$v_\delta(t', x') = v(t, x), \qquad x' \in B'', \qquad t' > -\delta^3,$$

$$v_\delta(t, x) = 0, \qquad x \in E - B,$$

where x' is the function of x and δ introduced in Section 2, and

$$t' = \begin{cases} t & \text{for} \quad t > \delta, \\ t - (\delta - t)^3 & \text{for} \quad 0 \leq t \leq \delta. \end{cases} \tag{3.14}$$

This function $v_\delta(x, t)$ possesses continuous second derivatives with respect to the x_i and t for $x \varepsilon B''$ and $t > -\delta^3$. It is therefore possible, as in Section 2 and in [2], to apply Taylor's formula with quadratic terms to $v_\delta(t, y)$. An application of Assumptions B and L yields, similarly as in [1] and in the proof of Theorem 3,

$$\int_E v_\delta(t, y)\, dF(y, x, \mu) = v_\delta(t, x) + \mu L[v_\delta] + \mu g_1(t, x, \mu, \delta), \qquad x \varepsilon B, \tag{3.15}$$

where the function g_1 has the property that for every $\epsilon > 0$, $\delta > 0$, $t_1 > 0$, a $\mu_0 > 0$ can be found such that

$$| g_1(t, x, \mu, \delta)| \leq \epsilon, \qquad x \varepsilon B + C, \qquad 0 \leq t \leq t_1, \qquad \mu \leq \mu_0 . \tag{3.16}$$

Now by the definition of v_δ it is possible to choose δ so small, independently of the value of μ, that

$$L[v_\delta] = \frac{\partial v_\delta}{\partial t} + g_2(t, x, \delta),$$

where $g_2(t, x, \delta)$ satisfies the same inequality as $g_1(t, x, \mu, \delta)$. Hence

$$\mu L[v_\delta] = v_\delta(t + \mu, x) - v_\delta(t, x) + \mu g_3(t, x, \mu, \delta), \tag{3.17}$$

where, for a certain positive $\mu_1 \leq \mu_0$, depending on δ and ϵ,

$$g_3(t, x, \mu, \delta) \leq 2\epsilon, \qquad x \varepsilon B + C, \qquad 0 \leq t \leq t_1, \qquad \mu \leq \mu_1 . \tag{3.18}$$

Combining (3.17) and (3.15) we find

$$v_\delta(t + \mu, x) = \int_E v_\delta(t, y)\, dF(y, x, \mu) + \mu h(t, x, \mu, \delta), \qquad x \varepsilon B, \tag{3.19}$$

where

$$| h(t, x, \mu, \delta)| \leq 3\epsilon, \qquad x \varepsilon B + C, \qquad 0 \leq t \leq t_1, \qquad \mu \leq \mu_1 . \tag{3.20}$$

Subtraction of (3.19) from (3.3) yields for

$$\omega(t, x, \mu) = P(t + t_0, x, \mu) - v_\delta(t, x) \tag{3.21}$$

the integral equation problem

$$\omega(t + \mu, x, \mu) = \int_E \omega(t, y, \mu)\, dF(y, x, \mu) + \mu h(t, x, \mu, \delta), \qquad x \varepsilon B, \quad t > 0, \tag{3.22}$$

$$\omega(t, x, \mu) = \omega_1(t, x, \mu), \qquad x \varepsilon E - B, \tag{3.23}$$

$$\omega(0, x, \mu) = \omega_2(x, \mu). \tag{3.24}$$

Here

$$| \omega_1(t, x, \mu)| = | P(t + t_0, x, \mu) - v_\delta(t, x)| = | 1 - v_\delta(t, x)| \leq \epsilon \tag{3.25}$$

for $x \varepsilon E - B$, $0 \leq t \leq t_1$, provided δ has been chosen sufficiently small (independently of μ). This follows from (3.13) and the continuity properties of $v_\delta(t, x)$. Similarly we have, if δ and μ_1 are, independently of each other, chosen sufficiently small,

$$
\begin{aligned}
|\omega_2(x, \mu)| = |P(t_0, x, \mu) - v_\delta(0, x)| &\leq |P(t_0, x, \mu) - Q(t_0, x)| \\
&+ |v(0, x) - v_\delta(0, x)| \leq \epsilon, \qquad x \varepsilon B + C.
\end{aligned} \tag{3.26}
$$

If we set $t = k\mu$ and write

$$\omega(k\mu, x, \mu) = u_k(x, \mu),$$

we can apply Lemma 5 to formulas (3.21) to (3.26) with the result that, if δ is sufficiently small,

$$|P(t + t_0, x, \mu) - v_\delta(t, x)| \leq 4C\epsilon,$$

for $x \varepsilon B + C$, $0 \leq t \leq t_1$, and $\mu \leq \mu_1$.

Finally, since v_δ differs arbitrarily little from v for sufficiently small δ, and ϵ was arbitrary, it follows that $Q(t, x) = \lim_{\mu \to 0} P(t, x, \mu)$ is, for all $x \varepsilon B$, the solution of the differential equation problem (3.11) to (3.13). By Lemma 4, $Q(t, x)$ approaches its values on C and its initial values for $t = 0$, as x approaches C or $t \to 0$, respectively, and is, therefore, indeed the solution of problem (3.5).

4. Some applications. If $L[u]$ is self-adjoint, then the solution of (2.6) can be calculated in the usual way by expansion in terms of the orthonormal eigenfunctions $u_j(x)$ of $L[u] + \lambda u = 0$, corresponding to the eigenvalues $\lambda = \lambda_j$, which are all real and positive. To do this we set $u = w + 1$ in (2.6) and solve the resulting problem

$$L[w] + sw = -s \text{ in } B, \; v = 0 \text{ on } C,$$

by the standard methods. (Cf., e.g., [9], p. 312. The argument for ordinary differential equations given there can be extended to partial differential equations whenever the existence of Green's function is known.) We find

$$
\begin{aligned}
u(s, x) = 1 + w(s, x) &= 1 + \sum_{j=1}^{\infty} \frac{s}{\lambda_j - s} \int_B u_j(y)\, dy \cdot u_j(x) \\
&= 1 + \sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\lambda_j - s} - 1 \right) \int_B u_j(y)\, dy \cdot u_j(x).
\end{aligned} \tag{4.1}
$$

The series $\sum_{j=1}^{\infty} \int_B u_j(y)\, dy \cdot u_j(x)$ is the generalized Fourier series of the function that is identically one in B. If this series actually converges to 1 in the interior of B, formula (4.1) simplifies to

$$u(s, x) = \sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j - s} \int_B u_j(y)\, dy \cdot u_j(x), \qquad x \varepsilon B. \tag{4.2}$$

From here on we assume explicitly that (4.2) is valid:

ASSUMPTION C. *The series* $\sum_{j=1}^{\infty} \int_B u_j(y)\, dy \cdot u_j(x)$ *converges for all* $x \varepsilon B$.

In this case we can give an explicit expression for the distribution function $Q(t, x)$, for

$$u(s, x) = \int_0^{\infty} e^{st} \sum_{j=1}^{\infty} \lambda_j \int_B u_j(y)\, dy \cdot u_j(x) e^{-\lambda_j t}\, dt,$$

and therefore, because of (3.4),

$$Q(t, x) = 1 - \sum_{j=0}^{\infty} \int_B u_j(y)\, dy \cdot u_j(x) e^{-\lambda_j t}. \tag{4.3}$$

This proves

THEOREM 6. *If the Assumptions B, L, E, S, and C are satisfied, if* $L[u]$ *is self-adjoint, and if the lowest eigenvalue* λ_1 *of* $L[u] + \lambda u = 0$ *is simple, then*

$$Pr\{N_x \geq k\} = u_1(x) \int_B u_1(y)\, dy \cdot e^{-\lambda_1 k\mu} + O(e^{-(\lambda_1 - \lambda_2) k\mu}) + \alpha(k\mu, x, \mu), \tag{4.4}$$

where

$$\lim_{\mu \to 0} \alpha(t, x, \mu) = 0$$

uniformly in t and x.

The leading term in (4.4) is thus a good approximation to $Pr\{N_x \geq k\}$ in a range of the variables μk for which $k\mu$ is so large, and at the same time μ so small, that the two remainder terms can be neglected.

The preceding calculations have some points of contact with those of M. Kac in [10], Section 10. The results there refer to the special case of Brownian motion. Also, an integral equation is used instead of (2.6), which permits a considerable relaxation of the condition S.

As a special application we consider random walks for which $L[u]$ reduces to a constant multiple of the Laplacian. It can then be assumed without loss of generality (see [1], Section 4) that μ is the mean square of the step length and that

$$L[u] = \frac{1}{2n} \Delta u. \tag{4.5}$$

A domain B for which all quantities involved can be calculated explicitly is the n-dimensional sphere of radius a with center at $x = 0$. A routine calculation leads to the formula

$$u(s, x) = \left(\frac{r}{a}\right)^{1-n/2} J_{n/2-1}(\sqrt{2nsr}) / J_{n/2-1}(\sqrt{2nsa}) \tag{4.6}$$

for the moment generating function. Here $r = (\sum_{j=1}^{n} x_j^2)^{\frac{1}{2}}$ and $J_k(z)$ is Bessel's function of the first kind. The series for $u(s, x)$ in powers of s,

$$u(s, x) = 1 + (a^2 - r^2)s + \frac{a^2 - r^2}{2n + 4}[(n + 4)a^2 - nr^2]s^2 + \cdots,$$

and an application of Theorems 3 and 4 lead to the expressions

$$\lim_{\mu \to 0} \mu E[N_x] = a^2 - r^2,$$

$$\lim_{\mu \to 0} \mu^2 E[N_x - E[N_x]] = \frac{2}{n + 2}(a^4 - r^4)$$

for the mean and the variance of the duration. The relative error $e[N_x]$, i.e., the standard deviation of N_x divided by its mean value, satisfies therefore the relation

$$\lim_{\mu \to 0} e[N_x] = \sqrt{\frac{2}{n + 2}} \sqrt{\frac{a^2 + r^2}{a^2 - r^2}}. \tag{4.7}$$

It should be noted that *the relative error is a decreasing function of the number of dimensions.*

We omit the straightforward calculation needed for the determination of the eigenfunctions $u_j(x)$ and eigenvalues λ_j in the present case and state only the results:

Let $\rho = \rho_j$ be the jth positive zero, in order of increasing size, of the function $J_{n/2-1}(\rho)$; then

$$\lambda_j = \rho_j^2/2na^2.$$

Assumption C is satisfied (cf., e.g., [11], p. 591) and

$$Q(t, x) = 1 - 2\left(\frac{r}{a}\right)^{1-n/2} \sum_{j=1}^{\infty} \frac{J_{n/2-1}(\rho_j r/a)}{\rho_j J_{n/2}(\rho_j)} e^{-\rho_j^2 t/2na^2} \tag{4.8}$$

For $n = 2$, we find, e. g., using the approximation (4.4), for small μ, and Theorem 4 with $k = 1$,

$$Pr\{N_0 \geq 2E[N_0]\} \sim 1/11. \tag{4.9}$$

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