DISTRIBUTION OF THE ORDINAL NUMBER OF SIMULTANEOUS EVENTS WHICH LAST DURING A FINITE TIME[1]

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1. Introduction. The probability of drawing a white ball from an urn is p, and the complementary probability of getting a black ball is $(1 - p) = q$. One ball is drawn and returned during one time unit. When a white ball appears, the play is interrupted for k time units. Then it starts anew.

If it happens that at the nth time unit a white ball occurs, we ask for the probability $w(m; n, k, p)$ that it is the mth ball since the first beginning of the trials. We are interested in the mean $E(m)$ and in the variance $\mathrm{Var}(m)$, and in simple approximations for $E(m/n)$ and $\mathrm{Var}(m/n)$ when n is large.

2. The probability distribution. Let us start with the relative probabilities. If the first white ball appeared at the nth moment, $(n - 1)$ black balls preceded, which means that the relative probability is q^{n-1}. If it was the second white ball, the number of black balls was reduced by $(k + 1)$, k for one interruption of the play lasting k time units and 1 for the first white ball, which occurred with the probability p. The group of $[(n - 1) - (k + 1)]$ black balls may be broken into any two parts, including the case of one being empty. That makes $\binom{(n-1)-k}{1}$ possibilities. Therefore the relative probability for $m = 2$ is

$$q^{n-1}\binom{(n-1)-k(m-1)}{m-1}\left(\frac{p}{q^{k+1}}\right)^{m-1}, \tag{1}$$

with $m = 2$. It is easy to verify step by step that the general formula is correct for $m = 1, 2, \cdots, 1 + \left[\frac{n-1}{k+1}\right]$. Hence the preliminary answer to our problem is

$$w(m; n, k, p) = \frac{1}{C} q^{n-1}\binom{(n-1)-k(m-1)}{m-1}\left(\frac{p}{q^{k+1}}\right)^{m-1}, \qquad k = 0, 1, 2, \cdots, \qquad m = 1, 2, \cdots, 1 + \left[\frac{n-1}{k+1}\right], \tag{2}$$

where $[a]$ means the largest positive integer $\leqq a$. The constant C has to be determined by

$$\sum_{m=1}^{1+[(n-1)/(k+1)]} w(m; n, k, p) = 1. \tag{3}$$

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[1] Opinions or conclusions contained in this paper are those of the author. They are not to be construed as necessarily reflecting the views or endorsement of the Navy Department.

This is a strictly algebraic problem. According to (3)

$$(4)\quad Cq^{1-n} = \sum_{m=1}^{1+[(n-1)/(k+1)]} \binom{(n-1)-k(m-1)}{m-1}\left(\frac{p}{q^{k+1}}\right)^{m-1} = s_n\left(\frac{p}{q^{k+1}}\right).$$

For abbreviation let us write

$$(5)\quad x = p/q^{k+1}.$$

It can be proved easily that

$$(6)\quad s_{n+1}(x) - s_n(x) - xs_{n-k}(x) = 0.$$

This is a linear recursion formula which can be solved in principle. The particular solution in question must satisfy the conditions

$$(7)\quad s_1(x) = s_2(x) = \cdots = s_{k+1}(x) = 1.$$

The characteristic equation belonging to this linear recursion formula with constant coefficients reads

$$(8)\quad \gamma^{k+1} - \gamma^k - x = 0,$$

or

$$(9)\quad f(z) = z^{k+1} - qz^k - p = 0$$

using the abbreviation $z = q\gamma$. Then

$$(10)\quad f'(z) = z^{k-1}((k+1)z - kq).$$

It follows that $f'(z) = 0$ for $z = 0$ (multiplicity $k - 1$) and for $z = kq/(k+1)$ (multiplicity 1). We find $f(0) = -1 + q \neq 0$ if $q \neq 1$. The exception is irrelevant. On the other hand

$$f(kq/(k+1)) = -k^k(q/(k+1))^{k+1} - p.$$

For $0 < q < 1$ this expression is negative. Hence $f'(z) = 0$ and $f(z) = 0$ do not have common roots. That means that the roots of the characteristic equation (9) are different from each other.

Therefore a regular solution of the linear difference equation (6) exists; it is

$$(11)\quad s_n(x) = \sum_{i=1}^{k+1} \frac{\gamma_i^n}{(k+1)\gamma_i - k},$$

where $\gamma_1, \gamma_2, \cdots, \gamma_{k+1}$ are the $(k+1)$ different roots of the characteristic equation (8). Summarizing, we find according to (2), (4), (11)

$$(12)\quad w(m; n, k, p) = \binom{(n-1)-k(m-1)}{m-1}\left(\frac{p}{q^{k+1}}\right)^{m-1} \Big/ \sum_{i=1}^{k+1} \frac{\gamma_i^n}{(k+1)\gamma_i - k}.$$

3. The mean and variance. We may write

$$(13)\quad w(m; n, k, p) = \binom{(n-1)-k(m-1)}{m-1} x^{m-1} \Big/ s_n(x),$$

where $x = p/q^{k+1}$ and $s_n(x)$ is given by (4). It is obvious that

$$\frac{x \dfrac{d}{dx} s_n(x)}{s_n(x)} = E(m - 1), \tag{14}$$

$$\frac{x^2 \dfrac{d^2}{dx^2} s_n(x)}{s_n(x)} = E[(m - 1)(m - 2)].$$

Hence

$$E(m) = 1 + \frac{x \dfrac{d}{dx} s_n(x)}{s_n(x)}, \tag{15}$$

$$E(m^2) = -2 + 3E(m) + \frac{x^2 \dfrac{d^2}{dx^2} s_n(x)}{s_n(x)}. \tag{16}$$

The finite series $s_n(x)$ can be differentiated term by term. We use the operator

$$\frac{d}{dx} = \frac{d}{d\gamma_i}\frac{d\gamma_i}{dx} = \frac{1}{\gamma_i^{k-1}[(k + 1)\gamma_i - k]}\frac{d}{d\gamma_i}. \tag{17}$$

The evaluation of (15) and (16) is only a technical matter. Let me list the final results:

$$E(m) = 1 + \frac{p/q^{k+1}}{\sum_{i=1}^{k+1} \dfrac{\gamma_i^n}{(k + 1)\gamma_i - k}} \left\{(k + 1)(n - 1) \sum_{i=1}^{k+1} \frac{\gamma_i^{n-k+1}}{[(k + 1)\gamma_i - k]^3} - kn \sum_{i=1}^{k+1} \frac{\gamma_i^{n-k}}{[(k + 1)\gamma_i - k]^3}\right\}; \tag{18}$$

$$\begin{aligned} E(m^2) = {} & -2 + 3E(m) \\ & + \frac{(p/q^{k+1})^2}{\sum_{i=1}^{k+1} \dfrac{\gamma_i^n}{(k + 1)\gamma_i - k}} \left\{(k + 1)^2(n - 1)(n - k - 2) \sum_{i=1}^{k+1} \frac{\gamma_i^{n-2k+2}}{[(k + 1)\gamma_i - k]^5}\right. \\ & - k(k + 1)[2n^2 - (2k + 3)n + (k - 1)] \sum_{i=1}^{k+1} \frac{\gamma_i^{n-2k+1}}{[(k + 1)\gamma_i - k]^5} \\ & \left. + k^2 n(n - k) \sum_{i=1}^{k+1} \frac{\gamma_i^{n-2k}}{[(k + 1)\gamma_i - k]^5}\right\}. \end{aligned} \tag{19}$$

Now we find as usual $\mathrm{Var}(m) = E(m^2) - [E(m)]^2$. These formulas are exact, but highly theoretical, since the roots γ_i are unknown if $k > 4$, except $\gamma_1 = 1/q$. Also, if we knew the roots, the equations would be too complicated. There is an urgent need for convenient approximations.

Let us write the characteristic equation (9) as follows:

$$p(1/z)^{k+1} + q(1/z) = 1. \tag{20}$$

This relation is impossible for $|1/z| < 1$ and is satisfied for $|1/z| = 1$ only if $1/z = +1$. Therefore, we have found that $z_1 = 1$ (according to (10) $\gamma_1 = 1/q$) is the absolutely largest root of the characteristic equation. Powers of the form

$$(\gamma_i/\gamma_1)^\lambda, \qquad i = 2, 3, \cdots, (k+1), \tag{21}$$

will converge rapidly to zero for large λ. This fact produces the following approximations, true for large n and for $k = 0, 1, 2, \cdots$:

$$w(m; n, k, p) \sim (1 + kp)q^{n-1} \binom{(n-1) - k(m-1)}{m-1} \left(\frac{p}{q^{k+1}}\right)^{m-1}, \tag{22}$$

with $1 \leqq m \leqq 1 + \left[\frac{n-1}{k+1}\right]$;

$$E(m/n) \sim \frac{p}{1+kp} + \frac{1}{n}\left(1 - \frac{(k+1)p}{(1+kp)^2}\right); \tag{23}$$

$$\operatorname{Var}(m/n) \sim \frac{1}{n}\frac{pq}{(1+kp)^3}\left(1 - \frac{k+1}{n}\frac{1-kp}{1+kp}\right). \tag{24}$$

Equation (22) is presented here only with a warning that the accuracy may not be good enough for some purposes. But the approximations (23) and (24) are fair.

For large n

$$\frac{\operatorname{Var}(m/n)}{[E(m/n)]^3} \sim \frac{1}{n}\frac{1-p}{p^2}, \tag{25}$$

independently of k. This relation is essentially a consequence of dimensional considerations. In applications the first member of (25) may be known from observations. Then the second member appraises p. From $E(m/n) \sim \frac{p}{1+kp}$ the parameter k could be estimated.

The equations (22) through (25) answer all our questions in a convenient manner. The new class of distributions is related to the Bernoullian type. It is to be expected that it will become useful in many fields of applied probability theory.