THE FORMATIVE YEARS OF ABRAHAM WALD AND HIS WORK IN GEOMETRY

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In the fall of 1927, a man of 25 called at the Mathematical Institute of the University of Vienna. Since he expressed a predilection for geometry he was referred to me. He introduced himself as Abraham Wald. In fluent German, but with an unmistakable Hungarian accent, Wald explained that he had carried on most of his studies at the elementary and secondary school levels at home, mainly under the direction of his older brother Martin, a capable electrical engineer in Cluj (Kolozsvár, Klausenburg). He had just arrived in Vienna in order to study mathematics at the university. Geometry had interested him ever since he was fourteen. More recently he had been reading Hilbert's "Grundlagen der Geometrie" (Foundations of Geometry) and he saw possibilities for improving these foundations by omitting some postulates and weakening others. I suggested to Wald that he write up his results {5}[1] (one of his proofs was later incorporated into the seventh edition of Hilbert's book) and at the same time recommended some additional reading.

Wald enrolled in the university, but during the next two years Vienna did not see much of him. The system of complete freedom which at that time prevailed in the universities of Central Europe—a detrimental system for weak students—kept the gifted ones from wasting semesters on courses the content of which they could absorb in a few weeks of concentrated reading. Moreover, Wald had to serve in the Rumanian army.

It was not until February 1930 that he and I again had extended conversations. Then he came unexpectedly to hand me a manuscript which purported to contain the solution of a famous problem. It was a serious piece of work, but an error at the very end invalidated the result. Wald was visibly disappointed. But a few days later he returned to tell me that, during the last week, he had been sitting in on my lectures on metric geometry—the first university lectures he ever attended—and that he planned to follow this entire course. Moreover, he wanted to try his hand at some problem in this field. I had just introduced the "between" relation in metric spaces: The point q is between the points p and r if, and only if, $p \neq q \neq r$ and the three distances between the points satisfy the equality

$$d(p, q) + d(q, r) = d(p, r).$$

I asked Wald whether he would like to try to characterize this "betweenness" among the ternary relations in a metric space. Four weeks later he brought me

[1] References are listed in "The publications of Abraham Wald," pp. 29–33 of this issue.

the first draft of the solution which he subsequently published in the Mathematische Annalen {1}, {2}, {3}, {7}. At the same time he asked for another problem.

It seemed to me that Wald had exactly the spirit which prevailed among the young mathematicians who gathered together about every other week in what we called our Mathematical Colloquium; so I at once invited him to present his result there. Gödel and Nöbeling, Alt and Beer were among the regular participants in this Colloquium; Miss Taussky came whenever she was in Vienna; Čech, Knaster, and Tarski were frequent guests; and numerous students and visitors came from abroad, especially from the United States and Japan. It was in this stimulating atmosphere that Wald spent his formative years. In these colloquia he became familiar with important problems, and presented the remarkable solutions which he published in the *Ergebnisse eines Mathematischen Kolloquiums.*

Wald's second course at the university was on dimension theory. I had suggested that topology might be developed in spaces other than point sets. Instead of "points," "pieces" might be the undefined basic concept. Certain nested sequences of pieces might be called points. Wald succeeded in characterizing the nested sequences which should be so named {4}.

After this excursion into topology, Wald returned to metric geometry. In 1928, I had characterized the metric spaces congruent to subsets of the n-dimensional euclidean space or of the Hilbert space. Wald solved the corresponding problem for the n-dimensional complex space (in which each point is given by n complex coordinates) as well as for all indefinite spaces where the coordinates of the points are real, but the square of the distance between two points is given by an indefinite quadratic form rather than by the definite sum of squares which goes back to the law of Pythagoras {14}. An unpublished manuscript, "On abstract fields and metrics," has been found. The paper is a continuation of the note of Miss Taussky in Issue 6 (pp. 20–23) of the *Ergebnisse.*

These studies aroused in Wald an interest in determinants {8} and led him to the following discovery. Let S be a four-dimensional simplex. It has 10 sides and 10 triangular faces. Geometers had known for a long time that the volume of S is determined by the lengths of its 10 sides. Is this volume also determined by the areas of the 10 faces? Wald constructed two simplexes with equal faces but different volumes {9}.

At that time Wald also became interested in Steinitz's theorem on the sums of series of vectors—a generalization of Riemann's famous result that any not absolutely converging series of real numbers can, by a permutation of its terms, be made to converge toward any number. Steinitz's theorem states that the vectors of an n-dimensional space, toward which a series of vectors can (by a permutation of the terms) be made to converge, form a linear manifold. Wald gave a new proof of the theorem and extended it to spaces of infinitely many dimensions. Moreover he studied series of group elements {11}, {12}, {13}.

In order to enhance the analogy between the postulates for Lebesgue measure and the postulates I had formulated for dimension, Wald developed a characterization of L-measure among set functions in which he confined the additivity postulates to closed sets {17}. Let $\mu(S)$ be a set function which (as, e.g., Lebesgue's exterior measure) is defined for every subset of the euclidean n-space and satisfies the following conditions: (1) $\mu(S) \geq 0$ for every S; (2) $S' \subset S$ implies $\mu(S') \leq \mu(S)$; (3) $\mu(C_1) + \mu(C_2) = \mu(C_1 + C_2)$ for any two disjoint closed sets; (4) $\mu(\sum_{i=1}^{\infty} C_i) \leq \sum_{i=1}^{\infty} \mu(C_i)$ for every sequence of closed sets; (5) $\mu(I) = 1$ for any unit n-cube I. Then, for every L-measurable set, $\mu(S)$ is equal to the L-measure of S.

In later years, he and I often joked about the fact that he took only one more course at the university before getting his Ph.D. This third course dealt with the new development of projective and affine geometry based on the operations of joining and intersecting which Garrett Birkhoff several years later called lattice operations. As a result of his studies in this field, Wald took active part {6} in the discussions of this subject which G. Bergmann, Alt, Schreiber, and myself carried on in the colloquium.

Another favorite topic of discussion there was the idea of curvature. On this subject Wald did his masterpiece in the field of pure mathematics. By virtue of the triangle inequality

$$d(p, q) + d(q, r) \geqq d(p, r),$$

three points p, q, r of a metric space are always congruent to three points of the euclidean plane. Consider the circum-circle of these latter points and call the reciprocal of its radius the curvature of the points p, q, r. This curvature is zero if and only if one of the three points is between the other two. Now let A be an arc contained in a metric space. Its points need not be given by coordinates, and its shape is not necessarily described by equations or functions. All that is assumed is an ordered continuum with a distance defined for every pair of points. For this general situation, I defined the curvature of A at the point a as the number (if it exists) from which the curvature of any three points differs arbitrarily little, provided all three points are sufficiently close to a. Numerous theorems were proved about this general curvature of curves, and about modifications of this concept due to Alt and Gödel. But the main problem was, of course, the extension of the idea to higher-dimensional manifolds.

From the outset it had been clear that, on a surface, quadruples of points should be considered. But what number should be associated with a given quadruple of points of a metric space? Four congruent points in the euclidean space do not necessarily exist; and even if they do exist, the radius of their circum-sphere is of no particular significance. Wald considered spheres metrized by the lengths of the arcs of great circles. For positive k, let S_k denote the sphere of curvature k thus metrized; S_0 is the euclidean plane; for negative k, let S_k be the hyperbolic plane of curvature k. If four points of a metric space are given, what S_k contains four congruent points? The difficulty of the problem is illustrated by the fact that for some quadruples of points no such S_k exists, whereas for some other quadruples there is more than one such S_k.

Wald overcame these and other difficulties. He proved {18}, {19}, {22} that if S is a surface of the type studied in classical differential geometry, then for each point a there exists a number $\kappa(a)$ with the following property: in S, for every quadruple of points each of which is sufficiently close to a, there exists a congruent quadruple in an S_k of which the curvature k differs arbitrarily little from $\kappa(a)$. Moreover, he proved that $\kappa(a)$ is equal to the famous Gauss curvature of S at the point a. Thus he obtained a new and very natural way of introducing Gauss's curvature. Even if he had stopped at this point, his result would have been a remarkable achievement.

But here was the beginning of Wald's really great work. In the second part of his paper {22} he dropped the assumption that a surface of the type studied in classical differential geometry be given. He dispensed with the characterization of points by coordinates and of surfaces by equations or functions or parametrizations. Continuing the idea which I had used in the simpler case of arcs, he merely assumed a compact metric space S with the following properties: (1) S is what I had called convex; that is, for any two points p and r of S, there exists a point between p and r; (2) S, at every point a, has a curvature $\kappa(a)$ (the symbol used in Wald's sense, for Gauss's definition of curvature is obviously inapplicable in this general situation). The second property means that for any four points of S which are sufficiently close to a, there exist four congruent points on a sphere S_k where k differs arbitrarily little from $\kappa(a)$. From this simple assumption Wald deduced (1) that S is a surface; (2) that in this surface polar coordinates can be locally introduced; (3) that in terms of these coordinates the length is expressed as it is on the classical surface of differential geometry; (4) that, for each point a of S, the number $\kappa(a)$ is equal to the Gauss curvature at a of the classical surface created on S by the introduction of the polar coordinates.

I venture to predict that the theorem just stated will become a cornerstone in the geometry of the future. This development may not please the devotees of classical differential geometry, for the theorem reveals serious redundancies in their assumptions. The essential features traditionally *postulated* (that is, coordinates which characterize points, parametric representations of surfaces, and of course, the differentiability of functions) can be *derived*. In fact, they can be derived from the one simple assumption of a convex compact metric space which at every point admits a Wald curvature. This result should make geometers realize that (contrary to the traditional view) the fundamental notion of curvature does not depend on coordinates, equations, parametrizations, or differentiability assumptions. The essence of curvature lies in the general notion of a convex metric space and of a quadruple of points in such a space. Some day these simple notions will be recognized as an adequate foundation for those local geometric properties the study of which for the last 250 years has been monopolized by differential geometers with their complicated conceptual machinery.

At this point I must interrupt the story of Wald's work and insert a few remarks about his life. He received his Ph.D. in 1931. At that time of economic and incipient political unrest, it was out of the question to secure for him a position at the University of Vienna, although such a connection would certainly have been as profitable for that institution as for himself. Outside of the Colloquium, my friend Hahn was the only mathematician who knew Wald personally. No one else showed the slightest interest in his work. However, Wald, with his characteristic modesty, told me that he would be perfectly satisfied with any small private position which would enable him to continue his work in our Mathematical Colloquium. I remembered that my friend Karl Schlesinger, a well-to-do banker and economist, wished to broaden his knowledge of higher mathematics; so I recommended Wald to him.

Out of the association between these two men grew Wald's interest in the equations of economic production. I asked Schlesinger to present his formulation of the equations to the Colloquium. Subsequently Wald published papers {15}, {21} on these ideas in the *Ergebnisse*, the first publications in his long list of contributions to mathematical economics. They have become classics in the field. Here, for the first time, economic equations were not merely formulated. The number of equations was not merely compared with the number of unknowns. The equations were solved. It was Schlesinger's modification of the original equations of Walras and Cassel which made them soluble. Soon after, I recommended Wald to Oskar Morgenstern, then director of the Austrian Institute for Business Cycle Research (Konjunkturforschung), and Morgenstern gave him employment in the Institute.

At that time there occurred a second event which proved to be of crucial importance in Wald's further life and work. The Viennese philosopher Karl Popper, now professor at the London School of Economics, tried to make precise the idea of a random sequence, and thus to remedy the obvious shortcomings of von Mises' definition of collectives. After I had heard (in Schlick's Philosophical Circle) a semitechnical exposition of Popper's ideas, I asked him to present the important subject in all details to the Mathematical Colloquium. Wald became greatly interested and the result was his masterly paper on the self-consistency of the notion of collectives {29} in the *Ergebnisse*. He based his existence proof for collectives on a twofold relativisation of that notion.

Let M be a (finite or infinite) set of symbols, such as H(ead) and T(ail), or 1, 2, 3, 4, 5, 6, or the points inside of a given square of the plane. By a selection of nth order Wald means a function f_n associating with every ordered n-tuple of elements $m_1, \cdots, m_n$ of M a value $f_n(m_1, \cdots, m_n)$ which is either 0 or 1; by a selector (*Auswahlvorschrift*), a sequence $S = \{f_0, f_1, \cdots, f_n, \cdots\}$. A selector makes it possible to select from every sequence of elements $\{m_1, m_2, \cdots, m_n, \cdots\}$ of M a subsequence

$$S\{m_1, m_2, \cdots, m_n, \cdots\} = \{m_{i_1}, m_{i_2}, \cdots, m_{i_n}, \cdots\}$$

by the following procedure:

m_{i_1} is the first element such that $f_{i_1-1}(m_1, \cdots, m_{i_1-1}) = 1$,

$\vdots$

m_{i_k} is the first element after $m_{i_{k-1}}$ such that $f_{i_k-1}(m_1, \cdots, m_{i_k-1}) = 1$,
and so on. Clearly, the selected subsequence may be infinite, finite or even vacuous.

Now let there be given (1) a set $\mathfrak{S}$ of selectors (including the identity which associates every sequence with itself); (2) a set $\mathfrak{M}$ of subsets of M. Then Wald calls a sequence $\{m_1, \cdots, m_n, \cdots\}$ *an* ($\mathfrak{S}$, $\mathfrak{M}$)*-collective* if for every set M^* belonging to $\mathfrak{M}$ there exists an $\mathfrak{S}$-probability $P(M^*, \mathfrak{S})$ in the following sense: every selector S belonging to $\mathfrak{S}$ selects from $\{m_1, \cdots, m_n, \cdots\}$ a subsequence $S\{m_1, \cdots, m_n, \cdots\} = \{m_{i_1}, \cdots, m_{i_n}, \cdots\}$ such that the relative frequency of M^* among the initial segments of the subsequence [that is, the number of elements of M^* among $\{m_{i_1}, \cdots, m_{i_n}\}$ divided by n] converges to $P(M^*, \mathfrak{S})$ as $n \to \infty$.

The two parameters $\mathfrak{S}$ and $\mathfrak{M}$ must be given if collectives are to be discussed in a self-consistent way. Although Wald's relativisation restricts the original unlimited (but unworkable) idea of collectives, it is much weaker than the irregularity requirements of Copeland, Popper, and Reichenbach. In fact, it embraces these requirements as special cases.

It was through this work on collectives and a study of time series {24} undertaken at Morgenstern's suggestion that Wald became interested in the foundations of statistics. But he kept on working at geometric problems, and added interesting remarks {25}, {26} to my first applications of metric methods to the calculus of variations.

Meanwhile the political situation in Austria deteriorated from month to month. The *Ergebnisse* was criticized (with specific reference to Wald) for its large number of Jewish contributions just when I felt that we ought to honor that journal by making Wald co-editor. Issue 7 was edited by Gödel, Wald, and myself. But Issue 8 containing Wald's paper on collectives was destined to be the last of the series. Hahn was dead. Schlick had been assassinated. Viennese culture resembled a bed of delicate flowers to which its owner refused soil and light while a fiendish neighbor was waiting for a chance to ruin the entire garden. I left the country. A year later Hitler marched into Vienna. Schlesinger, who occupied a rather prominent position, chose death that same day. These events foreshadowed the fate which later overtook Wald's family to which he was deeply attached. His parents and his sisters were murdered in the gas chambers of Ossoviec (Auschwitz); his brother Martin, the engineer, perished as a slave laborer in Western Germany.

Wald himself continued for a few weeks after Hitler's arrival in Vienna. He was dismissed by Morgenstern's successor but not otherwise molested. But I was greatly worried about his future as long as he remained in Austria, and with other friends, I tried to get him to the United States. Thanks to his work in econometrics and statistics he was permitted to come. Economists and statisticians soon became aware of his potentialities, and from the outset he was gratified to feel that this country would make effective use of his talents and abilities.

When he ceased working in the field of geometry, it was not for lack of interest. It was for lack of time. Whenever he and I met during the summer (we usually spent our vacations together in the mountains) we discussed both geometry and statistics. Wald's last geometric papers, {53}, {63}, date from 1943. By a strange coincidence, they deal with the "between" relation to which his first publication was devoted—but on a different level. In 1942 I had introduced a statistical metric in which the distance between two points is a distribution function rather than a number. Wald improved my original triangle inequality, upheld the definition of betweenness by a triangle equality, and proved that, even on the statistical level, betweenness has the properties by which, in 1930, he had characterized it among the ternary relations in metric spaces.

I have often wondered what would have happened if Wald had continued his geometrical work. A safe conjecture is that, if he had returned to geometry, that subject would have been greatly benefited. He and I had planned to work out a new differential geometry and vector analysis. If we had succeeded, a metric theory of the curvature of higher-dimensional spaces would now be in existence.

Another probable conjecture is that his geometric work would not have found the acclaim accorded to his work in applied mathematics. Geometry is not fashionable today. Although it is bound to outlive some current ephemeral fashions, even Wald's powerful talent would probably not have turned the tide. He might have remained a great but relatively unknown geometer.

Be that as it may, what Wald actually accomplished in geometry is of the first importance. I realize the high value of his papers on econometrics and of his book on sequential analysis, and I am aware of the profound influence which his theory of decision functions is bound to exert for decades to come. But nevertheless I believe that anyone who really understands his theory of the curvature of surfaces will find that this work is second to none of Wald's other achievements.