A LOWER BOUND FOR A PROBABILITY MOMENT OF ANY ABSOLUTELY CONTINUOUS DISTRIBUTION WITH FINITE VARIANCE

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Summary. The greatest lower bound of the nth probability moment (1.1) of a population with variance σ^2 is given by (3.4).

1. Introduction. The nth probability moment of a population with the probability density function $f(x)$ is defined as

$$\Omega_n = \int_{-\infty}^{\infty} [f(x)]^n \, dx. \tag{1.1}$$

These functionals have drawn the attention of some authors (see for instance [1] and the references given there) in connection with fitting frequency curves by means of frequency moments. Also it is to be noted[1] that the cumulative distribution function of the range w of a sample of size n from such a population can be approximated, for small w, by $n\Omega_n w^{n-1}$.

In general, it is not necessary that n in (1.1) be an integer. It may be any real number. However we put the restriction $n > 1$ in the following. To be specific, we take the population mean equal to zero. Moreover, we consider only populations whose variance

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) \, dx \tag{1.2}$$

is finite.

As the probability density function, $f(x)$ must satisfy the conditions

$$\int_{-\infty}^{\infty} f(x) \, dx = 1, \tag{1.3}$$

$$f(x) \geqq 0. \tag{1.4}$$

Under these conditions, we try to find a lower bound for Ω_n.

Incidentally, Ω_n has no finite upper bound, because it increases indefinitely as, for instance, the probability concentrates more and more to a certain point.

2. Derivation of the extremal distribution. The calculus of variations suggests equating to zero the first variation

$$\delta\left[\int_{-\infty}^{\infty} [f(x)]^n \, dx - \lambda \int_{-\infty}^{\infty} x^2 f(x) \, dx - \mu \int_{-\infty}^{\infty} f(x) \, dx\right], \tag{2.1}$$

[1] The author in fact took up this problem at first in connection with his work on the distribution of sample ranges. It was Professor Harold Hotelling who called the author's attention to probability moments in this relation.

where λ and μ are the Lagrange multipliers. Thus we get as the characteristic equation

$$n[f(x)]^{n-1} - \lambda x^2 - \mu = 0, \tag{2.2}$$

whence

$$f(x) = \left[\frac{\lambda x^2 + \mu}{n}\right]^{1/(n-1)} \tag{2.3}$$

We should take λ negative, and consequently μ positive. Then the solution (2.3) is applicable in the interval $(-\sqrt{-\mu/\lambda}, \sqrt{-\mu/\lambda})$. Outside of the interval, $f(x)$ should be taken to be identically equal to zero.

TABLE 1

Reduced probability moment, $\Omega_n \sigma^{n-1}$

Order	Lower bound	Normal distribution	Rectangular distribution	Asymptotic formula
2	.26833	.28209	.28868	.2547
3	.07599	.09189	.08333	.0735
4	.02174	.03175	.02406	.0212
5	$.0^{2}6245$	.01133	$.0^{2}6944$	$.0^{2}613$
6	$.0^{2}1797$	$.0^{2}4125$	$.0^{2}2005$	$.0^{2}177$
7	$.0^{3}5174$	$.0^{2}1524$	$.0^{3}5787$	$.0^{3}511$
8	$.0^{3}1491$	$.0^{3}5686$	$.0^{3}1671$	$.0^{3}147$
9	$.0^{4}4299$	$.0^{3}2139$	$.0^{4}4823$	$.0^{4}426$
10	$.0^{4}1240$	$.0^{4}8094$	$.0^{4}1392$	$.0^{4}123$

As a change of scale in measuring x does not affect the result essentially, we take the nonvanishing interval to be $(-1, 1)$, and write the solution in the form

$$\begin{aligned} f(x) &= c(1 - x^2)^{1/(n-1)}, & -1 \leqq x \leqq 1, \\ &= 0, & |x| > 1, \end{aligned} \tag{2.4}$$

where c is determined by the normalizing condition (1.3) as

$$c = \frac{1}{B\left(\frac{n}{n-1}, \frac{1}{2}\right)}. \tag{2.5}$$

The variance of the distribution (2.4) is calculated as

$$\sigma^2 = \frac{n-1}{3n-1}. \tag{2.6}$$

The nth probability moment of the distribution (2.4) is

$$\Omega_n = \frac{2n}{3n-1} c^{n-1}. \tag{2.7}$$

Therefore the reduced nth probability moment $\Omega_n \sigma^{n-1}$, which is invariant under any linear transformation of x, is given for the distribution of the same type as (2.4) by

$$\Omega_n \sigma^{n-1} = \frac{2n}{3n-1}\left[\sqrt{\frac{n-1}{3n-1}} \Big/ \mathrm{B}\left(\frac{n}{n-1}, \frac{1}{2}\right)\right]^{n-1}. \tag{2.8}$$

That this value gives the lower bound for any population with finite variance is to be proved in the next section.

3. Proof that the solution gives the lower bound. Let us denote the particular probability density function (2.4) by $\bar{f}(x)$, and compare the probability moment $\bar{\Omega}_n$ for it with Ω_n for any distribution with probability density function $f(x)$ and the same variance σ^2. From the normalizing condition and the assumed equality of the variance, we get

$$\int_{-\infty}^{\infty} [f(x) - \bar{f}(x)]\, dx = 0, \quad \int_{-\infty}^{\infty} x^2[f(x) - \bar{f}(x)]\, dx = 0. \tag{3.1}$$

By virtue of these equations and taking account of (2.4), we can express the difference $\Omega_n - \bar{\Omega}_n$ in the following way:

$$\begin{aligned} \Omega_n - \bar{\Omega}_n &= \int_{-\infty}^{\infty} [\{f(x)\}^n - \{\bar{f}(x)\}^n - nc^{n-1}(1 - x^2)\{f(x) - \bar{f}(x)\}]\, dx \\ &= \int_{-1}^{1} [\{f(x)\}^n - \{\bar{f}(x)\}^n - n\{\bar{f}(x)\}^{n-1}\{f(x) - \bar{f}(x)\}]\, dx \\ &\quad + \int_{|x|>1} [\{f(x)\}^n + nc^{n-1}(x^2 - 1)f(x)]\, dx. \end{aligned} \tag{3.2}$$

But Taylor's expansion up to the second-order term provides the formula, for any f and $\bar{f}$,

$$f^n = \bar{f}^n + n\bar{f}^{n-1}(f - \bar{f}) + \frac{n(n-1)}{2} f_1^{n-2}(f - \bar{f})^2, \tag{3.3}$$

where f_1 is a value between f and $\bar{f}$. As both $f(x)$ and $\bar{f}(x)$ are positive, the formula (3.3) assures us that the integrand of the first integral in the last member of (3.2) is nonnegative. Also the integrand of the second integral is obviously nonnegative. Hence we get the conclusion $\Omega_n \geqq \bar{\Omega}_n$, equality being satisfied only if $f(x) \equiv \bar{f}(x)$.

In general, it is easily derived from the above and (2.8) that

$$\Omega_n \geqq \frac{2n}{3n-1}\left[\sqrt{\frac{n-1}{3n-1}} \Big/ \mathrm{B}\left(\frac{n}{n-1}, \frac{1}{2}\right)\right]^{n-1} \frac{1}{\sigma^{n-1}}. \tag{3.4}$$

Thus the lower bound of a probability moment of any absolutely continuous distribution with finite variance σ^2 is given by the right-hand member of (3.4). It is actually achieved by a distribution of the same type as (2.4).

4. Numerical results. Numerical values of the coefficient in (3.4) are tabulated in Table 1, together with the corresponding values $n^{-\frac{1}{2}}(2\pi)^{-\frac{1}{2}(n-1)}$ for normal and $(2\sqrt{3})^{-(n-1)}$ for rectangular population. All these values approach 1 when $n \to 1$, as might be expected from the fact that for any distribution $\Omega_1 = 1$. It is to be noted that the curve for the lower bound would be fairly parallel in logarithmic scale to the curve for rectangular population. In fact it is easily shown that when n becomes large the former is given by

$$\begin{aligned} &\frac{1}{(2\sqrt{3})^{n-1}}\frac{2}{3}\exp\left[-\frac{1}{3}+\psi\left(\frac{1}{2}\right)-\psi(0)\right]\left(1+O\left(\frac{1}{n}\right)\right) \\ &\qquad = \frac{1}{(2\sqrt{3})^{n-1}}\frac{e^{5/3}}{6}\left(1+O\left(\frac{1}{n}\right)\right) = \frac{0.8824}{(2\sqrt{3})^{n-1}}\left(1+O\left(\frac{1}{n}\right)\right), \end{aligned} \tag{4.1}$$

where $\psi(x)$ denotes the digamma function $\Gamma'(x+1)/\Gamma(x+1)$. The first term happens to be close to the true value even for small n as we see in Table 1.

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REFERENCE

[1] HERBERT S. SICHEL, "The method of frequency-moments and its application to type VII populations," *Biometrika*, Vol. 36 (1949), pp. 404–425.

UNIFORMITY FIELD TRIALS WHEN DIFFERENCES IN FERTILITY LEVELS OF SUBPLOTS ARE NOT INCLUDED IN EXPERIMENTAL ERROR

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1. Introduction. The present note is confined to the consideration of two randomized blocks with two subplots each. The usual mathematical model for the analysis of variance of such an experiment assumes that

$$v_{ij} = g + b_i + t_j + \epsilon_{ij}, \qquad i = 1, 2; j = 1, 2, \tag{1.1}$$

where v_{ij} is the yield of the jth variety in the ith block, and the block effect b_i is the average for the subplots of the ith block. Any difference between b_i and the yield of subplots due to differences in fertility is one component of the random parts, ϵ_{ij}. The random parts, ϵ_{ij}'s, are then assumed to be normally and independently distributed with *zero* means and uniform variance. That these assumptions may break down in many cases because of the magnitude and nonrandomness of the differences between subplots has been indicated in a recent paper [1]. It should be understood that it is practically impossible with our present knowledge to determine the relative or absolute fertility levels of any set of plots,