NOTES

AN ASYMMETRIC BELL-SHAPED FREQUENCY CURVE

By Fausto I. Toranzos

Mendoza, Argentina

1. Summary. Frequency functions of the form (1) below including Pearson's system as a special case are derived from a generating differential equation (2). Their properties are discussed, methods of fitting them are suggested, and their characteristic function is determined.

2. Introduction. We wish to contribute to the study of the frequency functions of the form[1]

$$y = f(x) \exp\{-\tfrac{1}{2}a^2 (x - b)^2\}. \tag{1}$$

We will realize this purpose by taking as a generating differential equation of our system of frequency functions

$$\frac{y'}{y} = \frac{Q_{m+1}(x)}{P_m(x)} \tag{2}$$

in which Q is a polynomial of degree $m + 1$ and P another of degree m. Carrying out the division of the polynomials of (2) gives

$$\frac{y'}{y} = \alpha x + \beta + \frac{Q_{m-1}(x)}{P_m(x)} \tag{3}$$

and integrating

$$y = k \exp\{\tfrac{1}{2}\alpha x^2 + \beta x + \varphi(x)\}, \qquad \varphi(x) = \int \frac{Q_{m-1}(x)}{P_m(x)}\, dx.$$

The integral $\varphi(x)$ can always be evaluated by decomposing it into integrals of the forms

$$\int \frac{dx}{(x - a)^r}, \qquad \int \frac{Mx + N}{(x^2 + px + q)^s}\, dx.$$

Bearing this in mind, it is easy to show that, if $\alpha < 0$ and the roots of the denominator are simple, y is equal to the Gauss function multiplied by Pearson's functions. When there are multiple roots, exponential factors will appear.

Pearson's system is a particular case of (3) when $\alpha = 0$, $\beta = 0$ and $m = 2$.

[1] In this work we generalize and complete the results that we have given in [1].

3. The case $m = 1$. The case $m = 1$ is particularly important. The corresponding differential equation is

$$\frac{y'}{y} = \frac{A_2 x^2 + A_1 x + A_0}{B_1 x + B_0}.$$

Making a change in the origin of x, we have

$$\frac{y'}{y} = \frac{a_2 x^2 + a_1 x + a_0}{x}, \tag{4}$$

and integrating,

$$y = k \exp\left\{\left(x + \frac{a_1}{a_2}\right)^2\right\} x^{a_0}.$$

This function gives bell-shaped curves when $a_2 < 0$, $a_0 > 0$. Now putting $a_2 = -a^2$, $a_1/a_2 = b$, $a_0 = c$, we have

$$y = kx^c \exp\left\{-\tfrac{1}{2}a^2 (x - b)^2\right\}. \tag{5}$$

The graph of y is a bell-shaped curve with zeros at the points $x_1 = 0$, $x_2 = \infty$, if we take as the range of x, $0 \leqq x \leqq \infty$. In this interval y has only one maximum.

This case if of great importance due to its possible applications in distributions of bell-shaped frequencies with a finite range to the left of the maximum and extending to infinity on the right. Such happens in the case of the distribution of relative percentage values of prices and volume, in which the decrease is limited (from 100% to 0%), whereas the increase is theoretically unlimited (from 100% on). In spite of the natural asymmetry, the curve is remarkably close to the Gauss curve in the vicinity of the maximum.

When $a_2 > 0$ and $a_0 < 0$ we have a U-shaped curve which has a minimum near $x = b$, and which becomes infinite at $x = 0$ and $x = \infty$.

When $a_2 < 0$ and $a_0 < 0$ we have a curve which has a zero at the point $x = \infty$, a minimum near the origin, and a maximum near $x = b$, and which becomes infinite at the origin.

When $a_2 > 0$ and $a_0 > 0$ the curve has a zero at the point $x = 0$, a maximum near $x = 0$, and a minimum near $x = b$, and becomes infinite at $x = \infty$.

4. Determination of the constants. In order to fit the function (5) to an empirically determined frequency series, we may use the method of moments in the same way as Pearson does for his system.

The differential equation (2) can be expressed as

$$xdy = y(a_2x^2 + a_1x + a_0).$$

Multiplying by x^s and integrating from 0 to ∞, we have

$$-(s + 1)m_s = a_2m_{s+2} + a_1m_{s+1} + a_0m_s,$$

in which m_s are moments of order s. Identifying the functional moments with the empirical moments and giving successively the values 0, 1, and 2 to s, we have a system of three equations with three unknowns from which we can determine a_2, a_1, a_0, and therefore a, b, and c.

We can also determine the constants by the least squares method using the equation

$$\ln y = \tfrac{1}{2}a_2 x^2 + a_1 x + a_0 \ln x + C.$$

The value of C should be adjusted so that it satisfies the condition

$$\int_0^\infty y\,dx = 1.$$

5. Determination of the characteristic function. The characteristic function $\phi(t)$ of (5) is defined by

$$\phi(t) = k\int_0^\infty x^c \exp\{-\tfrac{1}{2}a^2(x-b)^2 + ixt\}\,dx.$$

Multiplying out the square, and putting $ab = p$, $ax = u$ we have

$$\phi(t) = ka^{-(c+1)} \exp\{-\tfrac{1}{2}a^2b^2\} \int_0^\infty u^c \exp\left\{-\tfrac{1}{2}u^2 + \left(p + \frac{it}{a}\right)u\right\} du$$

$$(6) \qquad = ka^{-(c+1)} \exp\{-\tfrac{1}{2}a^2b^2\}\, L_{-(p+it/a)}\,[u^c \exp\{-\tfrac{1}{2}u^2\}]$$

$$= ka^{-(c+1)}\, \Gamma(c+1) \exp\{-\tfrac{1}{2}a^2b^2 - \tfrac{1}{4}(p + it/a)^2\}\, D_{-(c+1)}\,[-p - it/a],$$

writing L for the Laplace transform, and $D_s(z)$ for Weber's function of the parabolic cylinder [2].

This function can be expressed by Whittaker's confluent hypergeometric function, which has been tabulated [2].

To determine the value of K it is sufficient to make $t = 0$ in (6), giving

$$\phi(0) = 1 = ka^{-(c+1)}\, \Gamma(c+1) \exp\left\{-\tfrac{1}{2}a^2b^2 - \frac{p^2}{4}\right\} D_{-(c+1)}\,[-p],$$

and therefore

$$k = \frac{a^{(c+1)}}{\Gamma(c+1)} \exp\left\{\tfrac{1}{2}a^2b^2 + \frac{p^2}{4}\right\} \Big/ D_{-(c+1)}\,[-ab],$$

and finally

$$\phi(t) = \exp\left\{\tfrac{1}{2}ibt - \frac{t^2}{a^2}\right\} D_{-(c+1)}\,[-ab - it/a]/D_{-(c+1)}\,[-ab].$$

Now we can determine the moments and other characteristics of the function (5).

REFERENCES

[1] FAUSTO I. TORANZOS, "A system of frequency curves which generalizes that of Pearson," *Revista Fac. Ci. Econ. Univ. Cuyo*, Vol. 1 (1949), 7 pp. (Spanish).

[2] E. T. WHITTAKER AND G. N. WATSON, *A Course of Modern Analysis*, Cambridge University Press, 1927, pp. 347–349.