$S_n(x)$ goes through $b_i\} \cdot P\{S_n(x)$ stays in band for $x > b \mid F_j(x)$, $S_n(x)$ goes through b_i and is in band for $x < b\}$. However the first and third of the factors is the same for $j = 0, 1$, and the second is unity for $j = 1$, and therefore $P_0 \leqq P_1$. If $\lambda/\sqrt{N} > 1/N$ (which is necessary if the test is not always going to reject) then at least for height b_k,

$$P\{S_n(x) \text{ inside the band for } x < b \mid S_n(b) = b_k, F_0(x)\} < 1.$$

Thus the test is biased.

I would like to thank Professor D. A. Darling for pointing out the error.

ABSTRACTS OF PAPERS

(Abstracts of papers presented at the East Lansing meeting of the Institute, September 2–5, 1952)

1. An Extension of Massey's Distribution of the Maximum Deviation between Two Sample Cumulative Step Functions. (Preliminary Report.) Chia Kuei Tsao, Wayne University.

Let $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_m$ be the ordered observations of two random samples from populations having cumulative distribution functions $F(x)$ and $G(x)$ respectively. Let $S_n(x) = k/n$ where k is the number of observations of X which are less than or equal to x and $S'_m(x) = j/m$ where j is the number of observations of Y which are less than or equal to x. The statistics $d_r = \max \mid S_n(x) - S'_m(x) \mid$ (max over $x < x_r$) and $d'_r = \max \mid S_n(x) - S'\ (x) \mid$ (max over $x < \max(x_r, y_r)$) can be used to test the hypothesis $F(x) \equiv G(x)$. For example, using d_r we would reject the hypothesis if the observed value of d_r is significantly large. In this paper, the methods of obtaining the distributions of d_r and d'_r (for small size samples) are similar to that in Massey's paper, and several short tables for equal size samples are included. (Work supported by the Office of Naval Research.)

2. Polynomial Correlation Coefficients. W. D. Baten and J. S. Frame, Michigan State College.

In this paper is developed a formula for the correlation coefficient pertaining to predicting polynomials. It is shown, when the independent variates are approximately normally distributed, that the square of this correlation coefficient can be expressed as a finite sum involving the squares of the averages of the derivatives of the estimating polynomial, namely, $r^2 = \Sigma \overline{y^{(k)}}^2/k!$, where y represents the predicting polynomial. The proof is based upon manipulations of Bernoulli numbers.

3. Truncated Poisson Distributions. Paul R. Rider, Wright-Patterson Air Force Base and Washington University.

This paper gives a method for estimating the parameter of truncated Poisson distributions for which some of the data are missing, particularly those which are truncated at the lower end. Application to a number of actual distributions is discussed.

4. Frequency Distributions for Functions of Rectangularly Distributed Random Variables. Stuart T. Hadden, Socony-Vacuum Laboratories, Paulsboro, New Jersey.

The theory of rectangularly distributed random variables is presented. It is shown how such random variables can occur in a certain class of controlled experiments arising in the fields of physics, chemistry, and engineering. On the basis of rectangularly distributed random variables, arising in observations or process variables, frequency distributions are developed for quantities which are functions of such variables. The principal method used in deriving the frequency distributions is operationally by means of the Laplace transform. Example applications illustrate how such frequency distributions can be applied in the analysis of experimental variance.

5. On Truncated Rules of Action. (Preliminary Report.) Benjamin Epstein, Wayne University.

A rule of action of theoretical and practical interest in life testing can be described as follows: (a) Non-replacement. Start the life test with n items drawn from a population. Let an integer r_0 and truncation time T_0 be preassigned. By the nature of the experiment failures will occur in order. Let $X_{r_0,n}$ be the time when the r_0th ordered failure occurs. If $X_{r_0,n} < T_0$, stop the experiment at $X_{r_0,n}$ and take action I. If $X_{r_0,n} > T_0$, stop the experiment at T_0 and take action II. (b) Replacement. Same as non-replacement except that a failed item is replaced at once by a new item. The properties of this kind of rule are investigated in detail when the underlying pdf is of the form $(1/\theta)e^{-x/\theta}$, $x > 0$, a distribution of some interest in life testing. The distributions of r, the number of items destroyed before taking an action, and T, the length of the experiment, are obtained. In particular $L(\theta)$, the probability of taking action I (say), $E_\theta(r)$, and $E_\theta(T)$ are obtained. Some tables based on this theory are obtained. (Work supported by the Office of Naval Research.)

6. The Distribution of the Difference of Two Independent Chi-Squares. James Pachares, University of North Carolina.

As a special case of the problem of the distributions of quadratic forms being investigated by the author, let $T_n = X_n - Y_n$, where X_n and Y_n are independently and identically distributed with probability density function (pdf) $[\Gamma(n/2)]^{-1}e^{-u}u^{(n-2)/2}$, $u > 0$. If $f_n(t)$ denotes the pdf of T_n, then the following recurrence equation holds: $f_{n+4}(t) = \{(n+1)/(n+2)\} f_{n+2}(t) + \{1/[n(n+2)]\}t^2 f_n(t)$, $n = 1, 2, \cdots$. The exact distribution of T_n is derived. If $K_n(t)$ is the modified Bessel function of the second kind of order n, then $f_n(t) = \pi^{-\frac{1}{2}}[\Gamma(n/2)]^{-1} (|t/2|)^{(n-1)/2}K_{(n-1)/2}(|t|)$, $n = 1, 2, \cdots$. Recurrence relations between the cumulative distribution functions (cdf's) of T_n are established so that any cdf for odd n depends on $F_1(x)$, while any cdf for even n depends on $F_2(x)$, where $F_n(x) = Pr[|T_n| \leqq x]$. A method is given for evaluating $F_1(x)$ by a series, with bounds on the error committed by stopping with a given term. Upper and lower bounds for $F_n(x)$ are given. (Work sponsored by the Office of Naval Research.)

7. Partially Balanced Designs with Two Plots per Block. R. C. Bose, University of North Carolina, and K. R. Nair, University of North Carolina and Forest Research Institute, Dehradun, India.

In many experimental situations, the block size is compulsorily restricted to two, as in comparing treatments given to two halves of a leaf. Partially balanced designs requiring only a small number of replications and with m accuracies $m \leqq 4$ have been worked out. It has been noticed that the association schemes of any known partially balanced incomplete block design with block size greater than 2 will lead to a design of the same type with block size 2, but a larger number of replications.

8. Minimax Sampling and Estimation in Finite Populations. OM PRAKASH AGGARWAL, Stanford University.

Stratified and cluster sampling from a finite population is considered from Bayes and minimax point of view. The loss in estimating the mean is taken to be the cost of observations plus the squared error in the estimate of the mean. For stratified sampling with linear cost function, for instance, it is shown that the minimax sampling plan chooses $n_i = \left\{\sqrt{(N_i^2\sigma_i^2/c_i) + \frac{1}{4}}\right\}$ individuals at random from the ith stratum and uses the usual estimate $f = \Sigma_{i=1}^k N_i\bar{X}_i$ for estimating $\Sigma_{i=1}^k N_i\mu_i$, where k is the number of strata, and in the ith stratum, N_i denotes the total number of individuals, μ_i, σ_i^2, the mean and variance, c_i the cost of sampling per individual, $\bar{X}_i$ the sample mean, and $\{q\}$ the integer nearest to q.

9. Some Two Sample Tests on the Exponential Distribution. (Preliminary Report.) BENJAMIN EPSTEIN AND CHIA KUEI TSAO, Wayne University.

Let S_{1n_1} and S_{2n_2} be two random samples such that S_{in_i} is a sample of size n_i from a population having pdf $(1/\theta_i)\exp[-(x - A_i/\theta)]$ $(i = 1, 2)$. Let S_{ir_i} be the set of the first $r_i (r_i \leqq n_i)$ smallest observations in S_{in_i}. On the basis of S_{1r_1} and S_{2r_2}, various likelihood ratio tests about the parameters involved can be obtained. The likelihood ratio tests about the hypothesis $\theta_1 = \theta_2$ assuming either A_1 and A_2 known or unknown are reducible to the well-known F-test. The test criterion for the hypothesis $A_1 = A_2$, assuming θ_1 and θ_2 known, may be reduced to a random variable having an exponential distribution. The tests of the hypothesis that $A_1 = A_2$ assuming θ_1 and θ_2 unknown, are also reduced to F-tests. Finally the test of the hypothesis $A_1 = A_2$ and $\theta_1 = \theta_2$ is obtained for the special case $r_1 = r_2$. (Work supported by the Office of Naval Research.)

10. Efficiency of Estimators of the Mean of an Exponential Distribution Based Only on the rth Smallest Observation in an Ordered Sample. BENJAMIN EPSTEIN, Wayne University.

Let us assume that the lives of certain items are describable by a positive random variable X, whose pdf is $f(x; \theta) = (1/\theta)e^{-x/\theta}$, $x > 0$. A sample of size n is drawn, and we suppose that the observations become available in order. Let the experiment be terminated at $x_{r,n}$, the time of failure of the rth item. We raise the question: How much information is lost if we base our estimate of the unknown parameter θ only on $x_{r,n}$ instead of basing it on all the first r failure-times, $x_{i,n}$, $i = 1, 2, \cdots, r$? As reported recently the m. l. estimate based on the $x_{i,n}$ is given by $\hat{\theta}_{r,n} = U/r$ where $U = \Sigma_{i=1}^r x_{i,n} + (n - r)x_{r,n}$. This estimate is "best" in the sense that it is unbiased, minimum variance, efficient, and sufficient. It is shown that unbiased estimates of θ based on $x_{r,n}$ alone have high efficiencies ($\geqq .9$) relative to $\hat{\theta}_{r,n}$ for values of $r \leqq 2n/3$. For example, for $r = n/2$, $n =$ even integer, the efficiency $\geqq 2(\log 2)^2 = .9608$. Tables giving the unbiasing constants $\beta_{r,n}$ such that $E(\beta_{r,n}x_{r,n}) = \theta$, $\mathrm{Var}(\beta_{r,n}X_{r,n})$, and the efficiencies $\mathrm{Var}(\hat{\theta}_{r,n})/\mathrm{Var}(\beta_{r,n}X_{r,n})$ have been obtained for $n = 1(1)20(5)30(10)100$ and $r = 1(1)n$. (Work supported by the Office of Naval Research.)

11. On the Theory of Systematic Sampling. III. WILLIAM G. MADOW, University of Illinois.

It is shown that if the elements of the population are constants and the population is monotone then centered systematic sampling is more efficient than random start systematic sampling; and that if the elements of the population are random variables and the correlogram is monotone decreasing then centered systematic sampling is more efficient than random start systematic sampling while if the correlogram is monotone increasing the contrary is true.

12. The Power of Some Service Tests. LEO A. GOODMAN, University of Chicago.

George W. Brown and Merrill M. Flood have presented in an interesting paper ("Tumbler Mortality", *Jour. Am. Stat. Assn.*, Vol. 42 (1947), pp. 567–574) the results of an analysis of a service test that was used to determine which of two types of glass tumblers had a longer mean length of life when used in a particular cafeteria. At the end of each week, each broken tumbler was recorded and replaced by a new one of the same type. Another kind of service test is based on the procedure of replacing the tumblers in equal numbers; i.e., as many of type 1 as of type 2, even though they broke in unequal numbers. Still another kind of service test is based on the procedure of replacing each broken tumbler by a new one of the other type. The preceding two procedures suggested, may be performed using either weekly records, or only the final count (the latter is less powerful, but less work). The exact power of these service tests is computed under the assumption of constant risk. The asymptotic power is computed in the more general case (non-constant risk). The several service tests are compared. This information may be used by the experimenter to decide which one of these tests to perform, and when to conclude the test.

13. A Minimal Essentially Complete Class of Tests of a Simple Hypothesis Specifying the Mean of a Unit Rectangular Distribution. ALLAN BIRNBAUM, Columbia University.

For the problem of testing a simple hypothesis on the mean of a unit rectangular distribution, on the basis of n ($n \geqq 2$) observations, explicit characterizations of the minimal complete class and a minimal essentially complete class of tests are given. Examples of tests which are best against various classes of alternatives are given; it is shown that the test with highest power against alternatives far from the null hypothesis has minimum power against alternatives close to the null hypothesis.

14. Application of Random Walk Theory to a General Class of Sequential Decision Problems. (Preliminary Report.) G. E. ALBERT, University of Tennessee.

One of r decisions d_i, $i = 1, 2, \cdots, r$, is to be made concerning the conditional cdf $F(y \mid x)$ of y given x, x and y in a Euclidean space R, by the following sequential experiment. Assign $r + 1$ nonnegative functions $p_i(x)$, $i = 0, 1, 2, \cdots, r$, on R with $\Sigma_{i=0}^{r} p_i(x) \equiv 1$. Perform a random walk beginning at an arbitrary point x_0, with successive points x_j drawn from $F(x_{j+1} \mid x_j)$, $j = 0, 1, 2, \cdots$, and terminating as soon as one of d_i, $i = 1, 2, \cdots, r$, has been decided under the following rule: let d_0 denote the decision to continue experimentation after any step x_j of the walk; at each step x_j, $j = 0, 1, 2, \cdots$, one of the decisions d_i, $i = 0, 1, 2, \cdots, r$, is made with respective probabilities $p_i(x_j)$. Let $P_i(x)$ denote the probability of making the decision d_i, $i = 1, 2, \cdots, r$, as a result of a walk starting at x. It is shown that under certain mild restrictions $P_i(x) = p_i(x) + p_0(x) \int_R P_i(y)\, dF(y \mid x)$. Also, the moment generating function and the moments of the duration of the experiment satisfy integral equations of a similar type; see Wasow, "On the duration of random walks." *Annals of Math. Stat.*, Vol. 22 (1951), pp. 199–216, for a special case. Some methods of approximating the solutions of these integral equations are established. Application of the theory is illustrated by a discussion of the sequential probability ratio test of hypotheses

θ_1, θ_2 on the parameter θ of a general class of cdf $G(x; \theta)$ which possesses a sufficient statistic for the parameter.

15. Nonparametric Comparisons of Populations when Data Are Collected in Homogeneous Groups. FRANK J. MASSEY, JR., University of Oregon.

The method of comparing two populations when data are paired has been fairly widely studied; for example, the sign test or t-test on differences. This paper presents similar techniques for analyzing data which have been collected in groups of size larger than one from each of several populations. Comparisons of power curves are made for certain normal alternatives (Work sponsored by the Office of Naval Research.)

16. On the Reduced Moment Problem. SALEM H. KHAMIS, Statistical Office, United Nations.

Let $\Phi(x)$ and $\Psi(x)$, $a \leqq x \leqq b$, be any two distinct cumulative distribution functions which are continuous and differentiable solutions of the reduced moment problem $\mu_r = \int_a^b x^r d\alpha(x), r = 0, 1, 2, \cdots, 2n$. A proof is given of the inequality (1) $| \Phi(x) - \Psi(x) | \leqq K\rho_n(x)$, where $\rho_n(x) = -| \mu_{i+j} |/D_n(x), i, j = 0, 1, 2, \cdots, n, D_n(x)$ is the determinant obtained by bordering the determinant $| \mu_{i+j} |$ by the prefixed row $(0\ 1\ x\ x^2 \cdots x^n)$ and the corresponding column, and where $0 < K = AB/(A + B - AB) \leqq \text{Min}(A, B) \leqq 1$ with $0 < A = 1 + \text{l.u.b.}_{a \leqq x \leqq b}(-\Phi'(x)/\Psi'(x)) \leqq 1$ and $0 < B = 1 + \text{l.u.b.}_{a \leqq x \leqq b}(-\Psi'(x)/\Phi'(x)) \leqq 1$. Inequality (1) is an improvement of an earlier result by the same author (*Proceedings of the International Congress of Mathematicians*, 1950, Vol. I, p. 569) which is in turn an improvement upon the corresponding Tchebycheff inequality, i.e., without the constant K (Shohat and Tamarkin, *The Problem of Moments*, American Mathematical Society, 1943, p. 72). By a special differencing method it is shown that the magnitude of the determinant in the numerator of $\rho_n(x)$ is independent of the origin of the moments, and that the determinant in the denominator is expressible in terms of the moments about the origin x. A method is also given for constructing an infinite number of cumulative distribution functions defined over a finite interval and possessing equal moments up to any given order, making use of the properties of orthogonal polynomials. Inequality (1) is then applied to the special class of such cumulative distribution functions associated with the Legendre polynomials.

17. Canonical Partial Correlations. S. N. ROY AND J. WHITTLESEY, University of North Carolina.

Canonical partial correlations between a set of p and a set of q variates, after elimination of a third set of r variates, is obtained by considering the canonical correlations between a set of $(p + r)$ and a set of $(q + r)$ variates having r variates in common. Suppose S is a $(p + q + r) \times (p + q + r)$ p.d. covariance matrix partitioned into submatrices such that the first row is $S_{11}(:p \times p)$ $S_{12}(:p \times q)$ $S_{13}(:p \times r)$, the second row is $S'_{12}(:q \times p)$ $S_{22}(:q \times q)$ $S_{23}(:q \times r)$, and the third row is $S'_{13}(:r \times p)$ $S'_{23}(:r \times q)$ $S_{33}(:r \times r)$. Then the canonical partial correlations between the p set and the q set are given by the p nonnegative roots (all lying between 0 and 1) of the equation in θ:

$$| \theta(S_{11} - S_{13}S_{33}^{-1}S_{13}) - (S_{12} - S_{13}S_{33}^{-1}S'_{23})(S_{22} - S_{23}S_{33}^{-1}S'_{23})^{-1}(S'_{12} - S_{23}S_{33}^{-1}S'_{13}) | = 0.$$

Putting (i) $r = 0$, (ii) $p = 1$, (iii) $p = 1, q = 1$, and (iv) $p = 1, r = 0$, we have respectively (i) canonical correlations, (ii) multiple partial correlation, (iii) partial correlation, and (iv) multiple correlation.

18. A Useful Transformation in the Case of Canonical Partial Correlations. S. N. Roy, University of North Carolina.

If the distribution problem in Abstract 31 were to be tackled ab initio, that is, without assuming the distributions of canonical correlations, the following transformation would be very helpful: $X_1(:p \times n) = U_1(:p \times p) \times (D_{\sqrt{1-\theta}} D_{\sqrt{\theta}}) \times$ (a $2p \times n$ matrix whose first row is $L_1(:p \times n)$ and the second row is $L_2(:p \times n)) + V_1(:p \times r)L_4(:r \times n)$. Also $X_2(:q \times n) =$ (a $q \times q$ matrix partitioned into 4 submatrices such that the first row is $U_{21}(:\overline{q-p} \times p)\tilde{U}_{22}(\overline{q-p} \times \overline{q-p})$ and the second row is $U_{23}(:p \times p)U_{24}(:p \times \overline{q-p})) \times$ (a $q \times n$ matrix whose first row is $L_2(:p \times n)$ and the second row is $L_3(:\overline{q-p} \times n)) + V_2(:q \times r)L_4(:r \times n)$, and lastly $X_3(:r \times n) = \tilde{U}_3(:r \times r)L_4(:r \times n)$ where the $(p + q + r) \times n$ matrix X, which is the reduced matrix of observations is supposed to be partitioned into $X_1(:p \times n)$, $X_2(:q \times n)$ and $X_3(:r \times n)$ placed one below the other, D_a stands for a diagonal matrix with elements $(a_1, \cdots, a_p)$, θ is given by the equation in the above abstract, $\tilde{M}$ stands for any triangular matrix with upper right hand corner zero, and $L'(:n \times \overline{p + p + q - p + r}) \equiv (L_1' L_2' L_3' L_4')$ is subject to $LL' = I(p + q + r)$. This transformation for an X of rank $p + q + r$ can be shown to exist and could also be made one to one if (i) the θ's are distinct, and, say, (ii) the first row of U_1, the diagonal elements of $\tilde{U}_{22}$ and of $\tilde{U}_3$ are all taken to be positive. This will of course happen almost everywhere (in the sample space). Erasing X_3, $\tilde{U}_3$, L_4, V_1 and V_2 we have the case of canonical correlations.

19. Uniform Convergence of Distribution Functions. Emanuel Parzen, University of California, Berkeley.

We determine conditions under which uniform convergence in a parameter θ of sequences of characteristic functions implies uniform convergence in θ of the corresponding sequences of distribution functions, which may be univariate or multivariate. We then derive a uniform central limit theorem and a uniform weak law of large numbers for sequences of independent random variables whose distribution depends on θ. These results may be applied to obtain conditions for the uniform consistency and uniform asymptotic normality of maximum likelihood estimates to be compared with those given by A. Wald ("Asymptotically most powerful tests of statistical hypotheses," *Annals of Math. Stat.*, Vol. 12 (1941), p. 2).

20. Statistical Aspects of a Linear Programming Problem. D. F. Votaw, Jr., Yale University.

The Hitchcock-Koopmans transportation problem is to determine a most economical program of transporting a homogeneous product (e.g., oil) from origins to destinations. The amounts of the product at the origins and required at the destinations are given together with the cost of transporting a unit amount from any origin to any destination. This paper is concerned with the analogous problem arising when the costs are unknown parameters in a distribution from which a sample is available. An application of the analysis of variance is pointed out, and some results of synthetic sampling are presented. (Research sponsored by the Office of Naval Research.)

21. Maximum Likelihood Estimators and A Posteriori Distributions. J. Wolfowitz, Cornell University.

Let $f(x, \theta)$ be the frequency function at x of each of the independent chance variables $x_1, \cdots, x_n$, whose distribution depends upon the parameter θ. Let $g(\theta')$ be the a priori density function of θ at θ', and let $h(\theta' \mid x_1, \cdots, x_n)$ be the a posteriori density function of θ at θ', given $x_1, \cdots, x_n$. Under suitable regularity conditions on f and g, h is asymptotically normal, with mean $\hat{\theta}$ and variance $[nc(\hat{\theta})]^{-1}$, where $\hat{\theta}$ is the maximum likelihood estimate of θ from $x_1, \cdots, x_n$ and $c(\theta) = \int((\partial \log f(x, \theta))/\partial\theta)^2 f(x, \theta)\, dx$. Thus the influence of g disappears in the limit. The present result includes that of v. Mises (*Math. Zeit.*, Vol. 4 (1919)) for the binomial case, and those of Kolmogoroff (*Izvyestya Akad. Nauk SSSR, Ser. Mat.*, Vol. 6 (1942)) for the normal case.

22. Estimates and Asymptotic Distributions of Certain Statistics in Information Theory. (Preliminary Report.) John P. Hoyt, U. S. Naval Academy.

In "On information and sufficiency" (S. Kullback and R. A. Leibler, *Annals of Math. Stat.*, Vol. 22 (1951), pp. 79–86), the concepts of "information" (designated hereafter as "i") and "mean information per observation" (designated hereafter as "I") for discrimination between two hypotheses were defined and various properties of "I" were proved for the abstract case. In "An application of information theory to multivariate analysis" (S. Kullback, *Annals of Math. Stat.*, Vol. 23 (1952), pp. 88–102), certain applications of information theory were made to multivariate analysis but problems of estimation and distribution were not considered. In the present paper, the characteristic function of the distribution of "i" in a sample of n from a normal multivariate population is found and from this is derived the expected value and variance of "i". A sample estimate of n "I" is then considered assuming equality of means in the two populations and a known value of one of the variance-covariance matrices occurring in "I". Using unbiased estimates of the parameters occurring in the other variance-covariance matrix, the characteristic function of the distribution of the estimate is found and is then used to show that the estimate's asymptotic distribution is given by the chi-square distribution with $k(k + 1)/2$ degrees of freedom.

23. On Testing One Simple Hypothesis Against Another. Lionel Weiss, University of Virginia.

Given a sequence $(X_1, X_2, \cdots)$ of independently and identically distributed chance variables, H_0 is the hypothesis that the probability density function of each chance variable is $f_0(x)$, H_1 is the hypothesis that this function is $f_1(x)$. A "generalized sequential probability ratio test" is defined as the usual Wald sequential probability ratio test, except that constant limits are not necessarily used; in other words, after the i *th* observation is taken, accept H_0 if the probability ratio is not greater than B_i, accept H_1 if the ratio is not less than A_i, otherwise take another observation, where $0 \leqq B_i \leqq A_i$. Given any test T of H_0 against H_1, not using randomization, and such that the probability that T will terminate is 1 when either H_0 or H_1 is true, then under mild restrictions on $f_0(x)$ and $f_1(x)$ the following theorem holds: There exists a sequence $(G_1, G_2, \cdots)$ of generalized sequential probability ratio tests such that Pr(sample size, when using G_j and H_i is true, is no greater than n) $\geqq Pr$(sample size, when using T and H_i is true, is no greater than n) for all n, all integers j, and $i = 0, 1$; and also, as j approaches infinity, lim Pr(H_i will be accepted when it is true and G_j is used) exists and is not less than Pr(H_i will be accepted when it is true and T is used), for $i = 0, 1$. If T is a truncated test, a stronger theorem holds; there exists a generalized sequential probability ratio test, also truncated, enjoying the above advantages over T.

24. Extreme Value Theory for m-Dependent Stationary Sequences of Continuous Random Variables. Geof. Watson, University of Melbourne.

The distributions, and their limits as $N \to \infty$, of the order statistics of N successive observations in a sequence of independent continuous random variables with a common distribution function, are well known. The present paper considers the same problem for sequences governed by stationary m-dependent probability laws. A stationary sequence is called m-dependent if $P(x_0 \leqq k_0 \mid x_{-1} \leqq k_{-1}, \cdots) = P(x_0 \leqq k_0 \mid x_{-1} \leqq k_{-1}, \cdots, x_{-m} \leqq k_{-m})$. These distributions are found in the general case here and it is shown that, as $N \to \infty$, their limiting forms are the same as the distributions obtained in the case of independence provided max $\{P(x_i > k, x_j > k)/P(x > k)\} \to 0(k \to \beta, k \leqq \beta)$ and max $\{P(x_i \leqq k, x_j \leqq k)/P(x \leqq k) \to 0(k \to \alpha, k \geqq \alpha)$, where the maximum is taken for $i, j = 1, \cdots, m + 1, i \neq j$, and where (α, β) is the range of the random variables $x_t(t = \cdots, -1, 0, 1, \cdots)$. Either or both of α and β may be infinite. These latter conditions are shown to be satisfied in all stationary normal processes. Thus the results of this paper give the limiting distributions of the order statistics in a sample of successive observations from any normal stationary autoregressive process.

25. Sequential Tests and Estimates for Comparing Poisson Populations. ALLAN BIRNBAUM, Columbia University.

The problem of testing a hypothesis on $\gamma = \lambda_2/\lambda_1$ is considered, where λ_1, λ_2 are the means of two Poisson populations. It is shown that no nonsequential test of $H_0 : \gamma = \gamma_0$ against $H_1 : \gamma = \gamma_1$ can have size uniformly $\leqq \alpha$ and power uniformly $\geqq 1 - \beta(1 - \beta > \alpha)$; a simple sequential test (not of the Wald type) is given which has these requirements of size and power against one- or two-sided alternatives. The generalization to the problem of classifying γ into one of k intervals is indicated. Comparisons with the Wald sequential tests of $H_0 : \gamma = \gamma_0$ against one-sided alternatives and of $H_0 : \gamma = 1$ against two-sided alternatives are made. The latter one-sided tests are constructed by use of a simple sufficient condition for the existence of a sequential probability ratio test of a composite hypothesis $H_0 : \theta \varepsilon \omega_0$, against a composite alternative $H_1 : \theta \varepsilon \omega_1$, of size approximately α for all $\theta \varepsilon \omega_0$ and power approximately $1 - \beta$ for $\theta \varepsilon \omega_1$. Application of this condition to problems of comparing two populations with Koopman-form distributions also gives tests which include those given by Girshick ("Contributions to the Theory of Sequential Analysis. I," *Annals of Math. Stat.*, Vol. 17 (1946), pp. 123–143), and some new tests for comparing variances of two normal populations. Tests of equality of ratios of means of two pairs of Poisson populations are given.

26. Sequential Decision Problems in the Stationary Case. J. KIEFER, Cornell University.

Results of Wald and Wolfowitz ("Bayes' solutions of sequential decision problems," *Ann. Math. Stat.*, Vol. 21 (1950), pp. 82–99; also, Chap. 4 of Wald's *Statistical Decision Functions*, John Wiley and Sons, 1950) are generalized to the case where the chance variables are no longer assumed independent, but instead form a stationary process. Questions of measurability and existence, recurrence formulas, characterizations of Bayes' solutions, etc., are simplified by first considering only nonrandomized decision functions and by then using results of the same authors ("Two methods of randomization in statistics and the theory of games," *Ann. Math.*, Vol. 53 (1951), pp. 581–586) to extend the conclusions to randomized procedures. The essential difference between the independent case and, e.g., the stationary Markoff case, is that in the latter a Bayes' solution may depend at each stage on the last observation as well as on the a posteriori distribution. For example, a Bayes' solution for testing between two simple hypotheses in the Markoff case is characterized by two functions $B(x) \leqq A(x)$ (which under slight restrictions are continuous) which are used after m observations the last of which is x_m by comparing the probability ratio to $B(x_m)$ and $A(x_m)$. Unlike the independent case, the $B(x)$ and $A(x)$ cannot in general be replaced by constants independent of x; nor does every pair $B(x)$, $A(x)$ constitute a Bayes' solution relative to some weight function, cost, and a priori distribution (as does every pair B, A in the independent case); nor need a Bayes' solution possess the optimum property of the independent case.

27. Random Functions Satisfying Certain Linear Relations. II. SUDHISH G. GHURYE, University of North Carolina.

The particular case $k = 1$ of the problem mentioned in Part 1 is considered here in detail. Let $X(t)$ be a p-dimensional, real-valued random function, defined and continuous in probability for all t in an interval $[t_0, T]$. Further, let there exist a real-valued, $p \times p$ matric function $A(h)$, defined and continuous for $h > 0$, such that if we write $Y(k; h) = X(t_0 + kh) - A(h)X(t_0 + [k - 1]h)$, then for any positive h and any integer n $(nh \leqq T - t_0)$, $X(t_0)$, $Y(1; h)$, $\cdots$, $Y(n; h)$ are mutually independent. Then it is shown that $A(h)$ can be written in the form e^{Bh}, where B is a constant matrix, and that $X^*(t) = e^{-Bt}X(t)$ is a random function with independent increments (r.f.i.i.). It is also shown that if $Z(t)$ is any p-dimensional r.f.i.i. and $A(t)$ is a $p \times p$ matric function, continuous and of bounded variation, then the integral $\int_{t_0}^{t} A(v)\, dZ(v)$ exists as the unique limit-in-distribution of the sequence of approximating sums. From this, a one-to-one correspondence (in distribution) between the random functions $X(t)$ mentioned above and the random functions $\int_{t_0}^{t} e^{(t-v)B}\, dZ(v)$ is established.

28. Optimal Designs for Estimating Parameters. (Preliminary Report.) HERMAN CHERNOFF, Stanford University.

The following is a generalization of a result of Elfving (see "Optimum allocation in linear regression theory," *Annals of Math. Stat.*, Vol. 23 (1952), pp. 255–262). It is desired to estimate parameters $\theta_1, \theta_2, \cdots, \theta_s$. There is available a set of experiments which may be performed. The probability distribution of the data obtained from any of these experiments may depend not only on $\theta_1, \theta_2, \cdots, \theta_s$ but also on the nuisance parameters θ_{s+1}, $\theta_{s+2}, \cdots, \theta_k$. One is permitted to select a design consisting of n of these experiments to be performed independently. The repetition of experiments is permitted in the design. Then it can be shown that under mild conditions and for large n locally optimal designs may be approximated by selecting a set of $r \leqq k + (k - 1) + \cdots + (k - s + 1)$ of the experiments available and by repeating each of these r experiments in certain specified proportions. The criterion of optimality used is a natural one involving the information matrices of the experiments.

29. The Distribution of the nth Variate in Certain Chains of Serially Dependent Populations. L. V. TORALBALLA, Marquette University.

The following is a representative of the problems considered: Let $P_1, P_2, \cdots, P_n$ be a sequence of normally distributed populations, the first having a mean m_1 and variance σ_1^2, each population after the first having a mean $m_i = ax_{i-1} + b$, where x_{i-1} is a random value of the variate in P_{i-1} and a variance σ_i^2. One seeks the absolute distribution of the variate in P_n. In this particular case it is found that the absolute distribution of the variate in P_n is normal, with a mean $b\Sigma_0^{n-2} a^i + a^{n-1}m_1$ and variance $\Sigma_{i=0}^{n-1} c^{2i}\sigma_{n-i}^2$.

30. An Experimental Method for Obtaining Random Digits and Permutations. J. E. WALSH, U. S. Naval Ordnance Test Station, China Lake.

This paper presents an easily applied method for obtaining small numbers of random binary digits and random permutations. The procedure consists in flipping ordinary minted coins and combining the results of the flips in an appropriate manner. Digits and permutations obtained according to the method of this paper can be considered sufficiently random for any practical application. It appears likely that these digits and permutations are much more nearly random than most of those now available in printed tables. Moreover, any possibility of bias from misuse of tables is avoided. The method presented is particularly suitable for use with respect to experimental designs. Only a few random permutations are ordinarily required for a given experimental design.

31. Distribution of Canonical Partial Correlations. S. N. ROY, University of North Carolina.

By certain general arguments the distribution of canonical partial correlations in random samples of size $n + 1$ from a $(p + q + r)$ variate normal population ($p \leqq q, p + q + r \leqq n$) can be shown to be of the same form as that of canonical correlations in random samples of size $n + 1 - r$, and involves as parameters (on the non-null hypotheses) the p roots (all lying between 0 and 1) of the equation in θ.

$$| \theta(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}') - (\Sigma_{12} - \Sigma_{13}\Sigma_{33}^{-1}\Sigma_{23}')(\Sigma_{22} - \Sigma_{23}\Sigma_{33}^{-1}\Sigma_{23}')^{-1}(\Sigma_{12}' - \Sigma_{23}\Sigma_{33}^{-1}\Sigma_{13}') | = 0,$$

where the population co-variance matrix Σ (supposed to be p.d.) is partitioned in the same manner as the sample covariance matrix S of Abstract 17.

In the abstract "On judging all contrasts in the analysis of variance" by Henry Scheffé (*Annals of Math. Stat.*, Vol. 23 (1952), p. 477) the equation $\Sigma_1^k c_i = 0$ was printed incorrectly (due to a compositor's error) as $\Sigma_1^k c_i \theta_i = 0$ on line 5.

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of interest

Personal Items

Mr. Fred C. Andrews has been appointed a Research Associate in the Applied Mathematics and Statistics Laboratory, Stanford University, Stanford, California.

Edward W. Barankin, Assistant Professor at the Statistical Laboratory, University of California, Berkeley, has been promoted to Associate Professor. For the academic year 1952/53, Dr. Barankin will be on leave, working at the Institute for Numerical Analysis, Los Angeles.

Z. W. Birnbaum, who has been on leave from the University of Washington for the academic year 1951–1952 and had a visiting professorship in the Department of Statistics at Stanford University, has returned to resume his duties at the University of Washington.

Dr. K. A. Bush, formerly Associate Professor of Mathematics at State Uni-