# SOME RELATIONS AMONG THE BLOCKS OF SYMMETRICAL GROUP DIVISIBLE DESIGNS

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**1. Summary.** It is well known that if every pair of treatments in a symmetrical balanced incomplete block design occurs in $\lambda$ blocks, then every two blocks of the design have $\lambda$ treatments in common. In this paper it will be shown that a somewhat similar property holds for symmetrical group divisible designs. In the course of the investigation there will be introduced certain matrices which are of intrinsic interest.

**2. Introduction.** Some of the combinatorial properties of group divisible incomplete block designs were considered in [1]. Here we shall need the definition of group divisible designs and the three classes into which they fall. An incomplete block design with $v$ treatments each replicated $r$ times in $b$ blocks of size $k$ is said to be group divisible (GD) if the treatments can be divided into $m$ groups, each with $n$ treatments, so that the treatments belonging to the same group occur together in $\lambda_1$ blocks and the treatments belonging to different groups occur together in $\lambda_2$ blocks, $\lambda_1 \neq \lambda_2$. The three exhaustive and mutually exclusive classes into which the GD designs fall are as follows:

(a) Singular GD designs characterized by $r - \lambda_1 = 0$;
(b) Semi-regular GD designs characterized by $r - \lambda_1 > 0$, $rk - v\lambda_2 = 0$; and
(c) Regular GD designs characterized by $r - \lambda_1 > 0$, $rk - v\lambda_2 > 0$.

In this paper we shall study classes (b) and (c) for the symmetrical case, that is, the case when $r = k$, or equivalently, $b = v$.

**3. The incidence and structural matrices.** In [2] there was defined the structural matrix for balanced incomplete block designs. We now shall define the incidence matrix, and two structural matrices for *GD* designs.

Let us consider first the incidence matrix of a *GD* design,

$$N = \begin{bmatrix} n_{11} & \cdots\cdots & n_{1b} \\ \vdots & & \vdots \\ n_{v1} & \cdots\cdots & n_{vb} \end{bmatrix}, \tag{3.1}$$

where the rows represent treatments, the columns represent blocks, and $n_{ij} = 1$ or 0 according as the $i$th treatment does or does not occur in the $j$th block. From the conditions satisfied by the design it is easy to see that

$$\sum_{j=1}^{b} n_{ij} = r \qquad (i = 1, \cdots, v), \tag{3.2}$$

[1] This work was begun while the author was at the University of North Carolina.

and

$$\sum_{j=1}^{b} n_{ij} n_{uj} = \lambda_1 \quad \text{or} \quad \lambda_2, \tag{3.3}$$

according as the $i$th and $u$th treatments $(i \neq u)$ do belong or do not belong to the same group.

Throughout the paper let us adopt the convention that the treatments $n(w - 1) + 1, n(w - 1) + 2, \cdots, nw$ shall belong to the $w$th group $(w = 1, \cdots, m)$. Then

$$NN' = \begin{bmatrix} A & B \cdots B \\ B & A \cdots B \\ \cdot & \cdot \quad\ \ \cdot \\ \cdot & \cdot \quad\ \ \cdot \\ \cdot & \cdot \quad\ \ \cdot \\ B & B \cdots A \end{bmatrix}, \tag{3.4}$$

where the elements of the $nxn$ submatrix $A$ are $r$ in the principal diagonal and $\lambda_1$ elsewhere, and the elements of the $nxn$ submatrix $B$ are $\lambda_2$ everywhere. Of course $NN'$ contains $v = mn$ rows and columns.

Now choose any $t \leqq b$ blocks of the design. Let the submatrix of $N$ which corresponds to these $t$ blocks be denoted by $N_0$. Let $s_{ju}$ be the number of treatments common to the $j$th and $u$th chosen blocks $(j, u = 1, 2, \cdots, t)$. Then the $t \times t$ symmetric matrix

$$S_t^I = N_0' N_0 = (s_{ju}) \tag{3.5}$$

is defined to be the *intersection structural matrix of the t chosen blocks*. The $j$th row or column of $S_t^I$ corresponds to the $j$th chosen block and the successive elements of the $j$th row or column give the number of treatments which this block has in common with the 1st, 2nd, $\cdots$, $t$th chosen blocks.

We next shall consider another structural matrix. Let $s_{ju}^w$ denote the number of treatments from the $w$th group which blocks $j$ and $u$ have in common. Then

$$\sum_{w=1}^{m} s_{ju}^w = s_{ju}, \tag{3.6}$$

$$\sum_{w=1}^{m} s_{jj}^w = k. \tag{3.7}$$

Now consider the matrix

$$G_t = \begin{bmatrix} s_{11}^1 & s_{22}^1 & \cdots & s_{tt}^1 \\ s_{11}^2 & s_{22}^2 & \cdots & s_{tt}^2 \\ \cdot & \cdot & \cdots & \cdot \\ s_{11}^m & s_{22}^m & \cdots & s_{tt}^m \end{bmatrix} \tag{3.8}$$

and the product matrix

$$S_t^G = G_t' G_t, \tag{3.9}$$

where the element in the $j$th row and the $u$th column is the sum of products of the number of treatments which the $j$th chosen block and the $u$th chosen block contain from each group. We define $S_t^G$ as the *group structural matrix of the $t$ chosen blocks.*

**4. The characteristic matrix.** We shall define an analogue of the characteristic matrix which was developed for balanced incomplete block designs in [2]. For the remainder of the paper, except for the last section, we shall restrict our attention to the regular $GD$ designs.

Let the columns of $N$ be permuted so that the first $t$ columns correspond to the $t$ chosen blocks. Then let the incidence matrix be extended by adjoining $t$ new rows, so that the elements of the $j$th adjoined row are zero, except for the $j$th which is unity. We thus get

$$N_1 = \begin{bmatrix} \multicolumn{2}{c}{N} \\ I_t & 0 \end{bmatrix}, \tag{4.1}$$

where $I_t$ is the identity matrix of order $t$, and 0 is the $t \times (b - t)$ zero matrix. Then

$$N_1 N_1' = \begin{bmatrix} NN' & N_0 \\ N_0' & I_t \end{bmatrix}. \tag{4.2}$$

The evaluation of $| N_1 N_1' |$ leads to

$$| N_1 N_1' | = (rk)^{-t+1}(r - \lambda_1)^{v-t-m}(rk - v\lambda_2)^{m-t-1} \, | C_t | , \tag{4.3}$$

where the typical element of $C_t$ is

$$c_{ju} = (rk - v\lambda_2)(rk\delta_{ju} + \lambda_2 k^2) + (\lambda_1 - \lambda_2)\left(rk \sum_{w=1}^{m} s_{jj}^{w} s_{uu}^{w} - n\lambda_2 k^2\right), \tag{4.4}$$

where $\delta_{ju} = (r - \lambda_1 - k)$ or $-s_{ju}$, according as $j = u$ or $j \neq u$. The matrix $C_t$ is defined as the *characteristic matrix of the $t$ chosen blocks*. The $j$th row or the $j$th column of $C_t$ corresponds to the $j$th chosen block of the design.

We observe that the characteristic matrix is related to the two structural matrices as is described in the following theorem.

THEOREM 4.1. *For the regular GD designs there exists a (1–1) correspondence among the elements of the intersection structural matrix $S_t^I$, the group structural matrix $S_t^G$, and the characteristic matrix $C_t$. This correspondence is given by*

$$C_t = rk(rk - v\lambda_2)[(r - \lambda_1)I_t - S_t^I] + rk(\lambda_1 - \lambda_2)S_t^G + \lambda_2 k^2(r - \lambda_1)E_t ,$$

*where $E_t$ is the singular $t \times t$ matrix all of whose elements are unity.*

For the particular case when $r = k$, the value of $| N_1 N_1' |$ as given by (4.3) reduces to

$$| N_1 N_1' | = r^{-2(t-1)}(r - \lambda_1)^{v-t-m}(r^2 - v\lambda_2)^{m-t-1} \, | C_t | , \tag{4.5}$$

where the typical element of $C_t$ is

$$c_{ju} = r^2(r^2 - v\lambda_2)(\delta_{ju} + \lambda_2) + r^2(\lambda_1 - \lambda_2)\left(\sum_{w=1}^{m} s_{jj}^{w} s_{uu}^{w} - n\lambda_2\right). \tag{4.6}$$

We shall state an analogue of Theorem 3.1 of [2]. The proof is as for that theorem.

THEOREM 4.2. *If $C_t$ is the characteristic matrix of any set of $t$ blocks chosen from a regular GD design with parameters $v$, $b$, $r$, $k$, $m$, $n$, $\lambda_1$*, and $\lambda_2$, *then*

(i) $| C_t | \geqq 0$ *if* $t < b - v$,

(ii) $| C_t | = 0$ *if* $t > b - v$, *and*

(iii) $r^{-2(t-1)}(r - \lambda_1)^{v-t-m}(r^2 - v\lambda_2)^{m-t-1} \, | C_t |$ is a perfect integral square, if $t = b - v$.

**5. Inequalities on $s_{ju}$ for regular symmetrical designs.** Let $t = 1$. Then since the factor outside of $| C_1 |$ in (4.5) is positive, it follows from Theorem 4.2 that $| C_1 | = 0$. Hence, from (4.6),

$$r^2(\lambda_1 - \lambda_2)\left[\sum_{w=1}^{m} (s_{11}^{w})^2 - r^2 + v\lambda_2 - n\lambda_2\right] = 0. \tag{5.1}$$

Since $r^2(\lambda_1 - \lambda_2) \neq 0$,

$$\sum_{w=1}^{m} (s_{11}^{w})^2 = r^2 - v\lambda_2 + n\lambda_2 . \tag{5.2}$$

Now let $t = 2$. Since $c_{11} = c_{22} = 0$, it is necessary by Theorem 4.2 that $c_{12} = c_{21} = 0$. Hence from (4.6),

$$s_{12} = \lambda_2 + \frac{e}{(r^2 - v\lambda_2)} (\lambda_1 - \lambda_2), \tag{5.3}$$

where

$$e = \sum_{w=1}^{m} s_{11}^{w} s_{22}^{w} - n\lambda_2 .$$

From (5.2) and the observation that $s_{jj}^{w} \geqq 0$ ($j = 1, 2; w = 1, \cdots, m$), it follows that

$$-n\lambda_2 \leqq e \leqq r^2 - v\lambda_2 . \tag{5.4}$$

From (5.3) and (5.4) we obtain

THEOREM 5.1. *For a regular symmetrical GD design the number of treatments $s_{ju}$ common to two blocks satisfies the inequalities*

$$\lambda_2(r - \lambda_1)/(r^2 - v\lambda_2) \leqq s_{ju} \leqq \lambda_1 ,$$

*when $\lambda_1 > \lambda_2$. The inequalities are reversed when $\lambda_1 < \lambda_2$.*

**6. The block structure for regular symmetrical GD designs when $r^2 - v\lambda_2$ and $\lambda_1 - \lambda_2$ are relatively prime.** We need to consider the distribution of the treatments contained in an initial block $B_1$ among the other blocks. Let $n_j$ be the number of blocks among the remaining $(b - 1)$ blocks which have $j$ treatments in common with $B_1$. Then from the definition of the design we obtain

$$\sum_{j=0}^{k} n_j = b - 1 = v - 1, \tag{6.1}$$
$$\sum_{j=0}^{k} jn_j = r(k - 1) = r(r - 1).$$

Also consider $M = \sum_{j=0}^{k} j(j - 1)n_j$, which is twice the number of pairs of treatments of $B_1$ which lie among the other blocks. $M$ is given by

$$M = \sum_{w=1}^{m} s_{11}^{w}(s_{11}^{w} - 1)(\lambda_1 - 1) + \sum_{\substack{x,w=1 \\ x \neq w}}^{m} s_{11}^{x} s_{11}^{w}(\lambda_2 - 1). \tag{6.2}$$

From (3.7) and (5.2), since $r = k$,

$$\sum_{w=1}^{m} s_{11}^{w}(s_{11}^{w} - 1) = (n - 1)\lambda_1, \tag{6.3}$$

$$\sum_{\substack{x,w=1 \\ x \neq w}}^{m} s_{11}^{x} s_{11}^{w} = (m - 1)n\lambda_2. \tag{6.4}$$

Hence

$$M = (n - 1)(\lambda_1)(\lambda_1 - 1) + (m - 1)(n)(\lambda_2)(\lambda_2 - 1). \tag{6.5}$$

Now consider

$$B = \sum_{j=0}^{k} (j - \lambda_1)(j - \lambda_2)n_j. \tag{6.6}$$

From (6.1), (6.5), and (6.6) we obtain

$$B = 0. \tag{6.7}$$

Hence the following lemma.

LEMMA 6.1. *If for a regular symmetrical GD design $n_j$ denotes the number of blocks which have $j$ treatments in common with a given initial block, then*

$$B = \sum_{j=0}^{k} n_j(j - \lambda_1)(j - \lambda_2) = 0.$$

Now let $r^2 - v\lambda_2$ and $\lambda_1 - \lambda_2$ be relatively prime. It follows from (5.3) that $s_{12}$ cannot lie in the open interval $(\lambda_1, \lambda_2)$. Then every term of $B$ is positive or zero. But since $B = 0$, every term must be zero. We thus get

THEOREM 6.1. *If for a regular symmetrical GD design $r^2 - v\lambda_2$ and $\lambda_1 - \lambda_2$ are relatively prime, then any two blocks have either $\lambda_1$ or $\lambda_2$ treatments in common.*

We further observe that even if $r^2 - v\lambda_2$ and $\lambda_1 - \lambda_2$ are not relatively prime, it still may not be possible to choose the elements of $G_t$ of (3.8), subject to the restrictions of (3.7) and (5.2), such that $s_{ju}$ is integral, but is not $\lambda_1$ or $\lambda_2$. Consider, for example, the *GD* design with parameters $v = b = 45$, $r = k = 9$, $m = 3$, $n = 15$, $\lambda_1 = 3$, and $\lambda_2 = 1$. The highest common factor of $r^2 - v\lambda_2$ and $\lambda_1 - \lambda_2$ is 2. It is clear that the only positive integers which satisfy (3.7) and (5.2) are 1, 1, and 7. But then we must have either $\sum_{w=1}^{m} s_{jj}^{w} s_{uu}^{w} = 51$ or 15, which correspond respectively to $\lambda_1$ and $\lambda_2$.

Now assume that the condition of Theorem 6.1 is met, or more generally, that positive integers do not exist which meet the restrictions of (3.7), (5.2) and Lemma 6.1 and imply values of $s_{ju}$ other than $\lambda_1$ and $\lambda_2$. Then from (6.1) we obtain

$$\begin{aligned} n_{\lambda_1} + n_{\lambda_2} &= v - 1, \\ \lambda_1 n_{\lambda_1} + \lambda_2 n_{\lambda_2} &= r(r - 1), \end{aligned} \tag{6.8}$$

whence

$$\begin{aligned} n_{\lambda_1} &= n - 1, \\ n_{\lambda_2} &= (m - 1)n, \end{aligned} \tag{6.9}$$

so that with respect to any initial block $B_1$, there are $(n - 1)$ other blocks which have $\lambda_1$ treatments in common with it, and $(m - 1)n$ other blocks which have $\lambda_2$ treatments in common with it.

From (5.3) we see that

$$\sum_{w=1}^{m} s_{11}^{w} s_{jj}^{w} = r + (n - 1)\lambda_1 \tag{6.10}$$

implies that blocks 1 and $j$ have $\lambda_1$ treatments in common, and conversely. But then from (5.2) and (6.10), it follows that

$$\sum_{w=1}^{m} s_{11}^{w} s_{jj}^{w} = \sum_{w=1}^{m} (s_{11}^{w})^2, \tag{6.11}$$

which implies that $s_{11}^{w} = s_{jj}^{w}$, $(w = 1, \cdots, m; j = 2, \cdots, b)$. Hence, if blocks $B_1$ and $B_j$ have $\lambda_1$ treatments in common, and blocks $B_1$ and $B_u$ have $\lambda_1$ treatments in common, then $B_j$ and $B_u$ have $\lambda_1$ treatments in common. We thus have

THEOREM 6.2. *If for a regular symmetrical GD design $r^2 - v\lambda_2$ and $\lambda_1 - \lambda_2$ are relatively prime, then the blocks fall into m groups of n blocks each, which are such that any two blocks from the same group contain $\lambda_1$ treatments in common and any two blocks from different groups contain $\lambda_2$ treatments in common.*

As has been indicated above, this theorem could be stated somewhat more generally.

**7. The semi-regular class.** For this class $rk - v\lambda_2 = 0$, and hence the above theory does not apply. We shall give a simple example which demonstrates for small $v$ that there do sometimes exist solutions in which $s_{ju} \neq \lambda_1$ or $\lambda_2$ for some $j$ and $u$.

Consider the $GD$ design with parameters $v = b = 8$, $r = k = 4$, $m = 4$, $n = 2$, $\lambda_1 = 0$, and $\lambda_2 = 2$. One solution is

$$N^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

which has the property that the blocks break up into 4 groups of 2 blocks each, which are such that two blocks in the same group have zero treatments in common and any two blocks from different groups have 2 treatments in common.

Another solution is

$$N^{(2)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

which is such that any initial block has 1 treatment in common with each of three blocks, 2 treatments in common with each of three blocks, and 3 treatments in common with one block.

We shall now obtain inequalities for the number of treatments $s_{ju}$ in common to any two blocks of a symmetrical semiregular $GD$ design. Since for a semi-regular $GD$ design, $rk = v\lambda_2$, it follows that $r - \lambda_1 = n(\lambda_2 - \lambda_1)$, from which we obtain the following lemma.

LEMMA 7.1. *For a semi-regular GD design, it is necessary that* $\lambda_2 > \lambda_1$.

Now let $r = k$. Choose any two blocks and let the columns of $N$ be permuted so that the first two columns correspond to the chosen blocks. Then to $N$ affix $m$ new columns, the $w$th of which contains $(\lambda_2 - \lambda_1)^{\frac{1}{2}}$ in the rows which correspond to the treatments of the $w$th group, $(w = 1, \cdots, m)$, and zero elsewhere. Let the augmented matrix be denoted by $N_2$. Now form

$$N_3 = \begin{bmatrix} & N_2 & \\ I_2 & & 0 \end{bmatrix}, \tag{7.1}$$

where $I_2$ is the identity matrix of order 2 and 0 is the $2x(b + m - 2)$ matrix all of whose elements are zero. Then

$$| N_3N_3' | = (r + v\lambda_2 - \lambda_1)^{-1}(r - \lambda_1)^{v-3} | B_2 |, \tag{7.2}$$

where $B_2$ is a $2 \times 2$ matrix with elements

$$\begin{aligned} b_{11} = b_{22} &= (r + v\lambda_2 - \lambda_1)(-\lambda_1) + \lambda_2 r^2, \\ b_{12} = b_{21} &= (r + v\lambda_2 - \lambda_1)(-s_{12}) + \lambda_2 r^2. \end{aligned} \tag{7.3}$$

As for Theorem 4.2 it is necessary that $| N_3N_3' | \geqq 0$, and since the factor outside of $| B_2 |$ in (7.2) is positive, it is necessary that $| B_2 | \geqq 0$. Hence, the following theorem:

THEOREM 7.1. *For a symmetrical semi-regular GD design, the number of treatments common to two blocks, $s_{ju}$, satisfies the inequalities*

$$\lambda_1 \leqq s_{ju} \leqq \frac{2\lambda_2 r^2}{r + v\lambda_2 - \lambda_1} - \lambda_1 .$$

I wish to express my thanks to Professor R. C. Bose for suggesting this problem.

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