NOTES

THE MONOTONICITY OF THE RATIO OF TWO NONCENTRAL t DENSITY FUNCTIONS[1]

BY WILLIAM KRUSKAL

University of Chicago

1. Summary. The ratio of two different noncentral t density functions with the same number of degrees of freedom is strictly monotone, with sense depending on the relative values of the two noncentral constants.

2. Background. The ratio of two noncentral t density functions has arisen in several statistical connections. First, in the proof that the Student t-test is uniformly most powerful invariant, the ratio of a noncentral t density function to a central t density function arises. This is discussed by Lehmann ([4], chap. 4) who gives a proof of monotonicity.

Second, the same ratio arises in the study of sequential t-tests; a discussion of this is given by Arnold in [1].

Third, the case in which *both* numerator and denominator are noncentral t density functions arises in connection with a sequential test for (one-sided) fraction defective. A discussion of this is given by Rushton [5], and an earlier reference to the same sequential test appears in *Selected Techniques of Statistical Analysis* ([2], p. 83, footnote). In this case, as well as in that of the above paragraph, monotonicity of the ratio is of interest because it implies that at any stage of sampling the continue-sampling values of the natural test statistic—Student's t—form an interval.

The purpose of this note is to give a very simple proof of the monotonicity of such ratios. The method is similar to that used by Wald ([6], Section A.8.2).

3. Statement. The noncentral t density function with ν degrees of freedom and noncentral parameter δ is

$$\phi(t; \nu, \delta) = \frac{\Gamma(\nu + 1)}{2^{\frac{1}{2}(\nu-1)}\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}} \left(\frac{\nu}{\nu + t^2}\right)^{\frac{1}{2}(\nu+1)} e^{-\frac{1}{2}[\nu\delta^2/(\nu+t^2)]}\, Hh_\nu\left(\frac{-\delta t}{\sqrt{\nu + t^2}}\right) \tag{3.1}$$

where

$$Hh_\nu(x) = \int_0^\infty \frac{z^\nu}{\Gamma(\nu + 1)} e^{-\frac{1}{2}(z+x)^2}\, dz. \tag{3.2}$$

Received 5/11/53.

[1] Based on research supported by the Office of Naval Research at the Statistical Research Center, University of Chicago.

This quantity is the density function for $(U + \delta)/\sqrt{W/\nu} = t$ where U and W are independent random variables having respectively unit-normal and $\chi^2(\nu)$ distributions. (The noncentral t density function is readily derived from the joint distribution of U and W. It may be of interest to mention two minor misprints in the statement of this function by Johnson and Welch [3]. In their (2) the function should be divided by $\sqrt{\nu}$; and in their (3) a minus sign should appear in the exponent.)

If we consider two such density functions for the same ν but with $\delta = \delta_0$ and δ_1 respectively the natural logarithm of the ratio of the two is

$$\ln \frac{\phi(t; \nu, \delta_1)}{\phi(t; \nu, \delta_0)} = -\tfrac{1}{2} \frac{\nu}{\nu + t^2} (\delta_1^2 - \delta_0^2) + \ln Hh_\nu \left(\frac{-t\delta_1}{\sqrt{\nu + t^2}} \right) - \ln Hh_\nu \left(\frac{-t\delta_0}{\sqrt{\nu + t^2}} \right). \tag{3.3}$$

I shall prove the following

THEOREM. *If $\delta_0 \neq \delta_1$, (3.3) is strictly monotone, increasing if $\delta_0 < \delta_1$ and decreasing if $\delta_0 > \delta_1$.*

4. Proof. Replace the independent variable t by the following strictly increasing function of it:

$$u = \frac{t}{\sqrt{\nu + t^2}}, \qquad t = u \sqrt{\frac{\nu}{1 - u^2}} \tag{4.1}$$

so that $\nu + t^2 = \nu/(1 - u^2)$ and we may write (3.3) in the form

$$\begin{aligned} &-\tfrac{1}{2}(1 - u^2)(\delta_1^2 - \delta_0^2) + \ln \frac{\int_0^\infty z^\nu e^{-\frac{1}{2}(z-\delta_1 u)^2}\,dz}{\int_0^\infty z^\nu e^{-\frac{1}{2}(z-\delta_0 u)^2}\,dz} \\ &= -\tfrac{1}{2}(\delta_1^2 - \delta_0^2) + \ln \frac{\int_0^\infty z^\nu e^{-\frac{1}{2}z^2+u\delta_1 z}\,dz}{\int_0^\infty z^\nu e^{-\frac{1}{2}z^2+u\delta_0 z}\,dz}. \end{aligned} \tag{4.2}$$

Differentiate (4.2) with respect to u and observe that the stated theorem is equivalent to the statement that the sign of

$$\begin{aligned} &\int_0^\infty z^\nu e^{-\frac{1}{2}z^2+u\delta_0 z}\,dz \int_0^\infty \delta_1 z^{\nu+1} e^{-\frac{1}{2}z^2+u\delta_1 z}\,dz \\ &- \int_0^\infty z^\nu e^{-\frac{1}{2}z^2+u\delta_1 z}\,dz \int_0^\infty \delta_0 z^{\nu+1} e^{-\frac{1}{2}z^2+u\delta_0 z}\,dz \end{aligned} \tag{4.3}$$

is the same as the sign of $\delta_1 - \delta_0$. By rewriting each of the above terms as a double integral in (z_0, z_1) and combining, we see that the desired result is further equivalent to showing that the sign of

$$(4.4) \qquad \int_0^\infty \int_0^\infty (z_0 z_1)^\nu \, e^{-\frac{1}{2}(z_0^2+z_1^2)+u(\delta_0 z_0+\delta_1 z_1)}(\delta_1 z_1 - \delta_0 z_0)\, dz_0\, dz_1$$

is the same as the sign of $\delta_1 - \delta_0$.

From (4.4) the truth of the theorem is immediate if either δ_1 or δ_0 is zero, or if δ_0 and δ_1 have opposite signs. Now suppose for simplicity that both δ_0 and δ_1 are positive.

Rewrite (4.4) in the following way:

$$(4.5) \qquad \begin{aligned} &\iint_{\delta_1 z_1 > \delta_0 z_0 > 0} (z_0 z_1)^\nu \, e^{-\frac{1}{2}(z_0^2+z_1^2)+u(\delta_0 z_0+\delta_1 z_1)} \, (\delta_1 z_1 - \delta_0 z_0)\, dz_0\, dz_1 \\ &- \iint_{0 < \delta_1 z_1 < \delta_0 z_0} (z_0 z_1)^\nu \, e^{-\frac{1}{2}(z_0^2+z_1^2)+u(\delta_0 z_0+\delta_1 z_1)} \, (\delta_0 z_0 - \delta_1 z_1)\, dz_0\, dz_1 \end{aligned}$$

and make the following changes of variable:

First Double Integral	*Second Double Integral*
$z_0 = s_0/\delta_0$	$z_0 = s_1/\delta_0$
$z_1 = s_1/\delta_1$	$z_1 = s_0/\delta_1$

to obtain

$$(4.6) \qquad \begin{aligned} &(\delta_0 \delta_1)^{-\nu-1} \iint_{s_1 > s_0 > 0} (s_0 s_1)^\nu e^{+u(s_0+s_1)}(s_1 - s_0) \\ &\qquad \left[\exp\left(-\tfrac{1}{2}\left(\frac{s_0^2}{\delta_0^2} + \frac{s_1^2}{\delta_1^2}\right)\right) - \exp\left(-\tfrac{1}{2}\left(\frac{s_1^2}{\delta_0^2} + \frac{s_0^2}{\delta_1^2}\right)\right)\right] ds_0 \cdot ds_1 . \end{aligned}$$

Hence the desired conclusion would be implied by the result that the sign of

$$(4.7) \qquad \frac{s_0^2}{\delta_0^2} + \frac{s_1^2}{\delta_1^2} - \frac{s_1^2}{\delta_0^2} - \frac{s_0^2}{\delta_1^2}$$

is opposite to that of $\delta_1 - \delta_0$ so long as $s_1 > s_0 > 0$. But (4.7) may be written as

$$\left(\frac{1}{\delta_0^2} - \frac{1}{\delta_1^2}\right)(s_0^2 - s_1^2)$$

whose sign is that of $\delta_0 - \delta_1$. This completes the proof for every case except that of δ_0, δ_1 both negative. But this goes through with obvious minor modifications in (4.5) and the subsequent manipulations.

I should like to thank Charles M. Stein for helpful comments made after reading a draft of this note.

REFERENCES

[1] KENNETH J. ARNOLD, *Tables to Facilitate Sequential t-Tests*, National Bureau of Standards, Applied Mathematics Series 7, 1951, pp. v–xiii.

[2] W. ALLEN WALLIS, "Use of variables in acceptance inspection for percent defective,"

Selected Techniques of Statistical Analysis, chap. 1, Statistical Research Group, Columbia University, New York, McGraw-Hill Book Co., 1947.
[3] N. L. JOHNSON AND B. L. WELCH, "Applications of the noncentral t-distribution," *Biometrika*, Vol. 31 (1939–1940), pp. 362–389.
[4] ERICH LEHMANN, *Theory of Testing Hypotheses*, Mimeographed notes recorded by Colin Blyth, ASUC Bookstore, University of California, Berkeley, 1948–1949.
[5] S. RUSHTON, "On a sequential t-test," *Biometrika*, Vol. 37 (1950), pp. 326–333.
[6] ABRAHAM WALD, *Sequential Analysis*, John Wiley and Sons, Inc., New York, 1947.

AN EXTENSION OF THE BOREL-CANTELLI LEMMA

BY STANLEY W. NASH

University of British Columbia

1. Introduction. Consider a probability space $(\Omega, \mathfrak{F}, P)$ and a sequence of events $\{A_n\}$, $A_n \in \mathfrak{F}$, $n = 1, 2, \cdots$. The upper limiting set of the sequence is defined to be

$$\lim_{n\to\infty} \sup_{k\geq n} A_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

It is the event that infinitely many of the A_n occur. The purpose of this paper is to find necessary and sufficient conditions for $P(\lim \sup A_n) = 1$.

The general problem of finding the probability of an infinite number of a sequence of events occurring was considered by Borel [1], [2] and Cantelli [3]. In what follows we shall use the following notations. Let $\alpha_n = I(A_n)$, the indicator of the event A_n (or characteristic function of the set A_n), that is

$$\alpha_n = \begin{cases} 1 \text{ when } A_n \text{ occurs} \\ 0 \text{ when } A_n \text{ fails to occur.} \end{cases}$$

Let $P(A_n \mid \alpha_1\alpha_2 \cdots \alpha_{n-1})$ denote the conditional probability of the event A_n, given the outcomes of the previous $n - 1$ trials. When $n = 1$, the expression is taken to represent the unconditional probability $P(A_1)$. The 1912 Borel criterion stated:

If $0 < p_n' \leqq P(A_n \mid \alpha_1\alpha_2 \cdots \alpha_{n-1}) \leqq p_n'' < 1$ for every n, whatever be α_1, $\alpha_2, \cdots, \alpha_{n-1}$, then $\sum_{j=1}^{\infty} p_j'' < \infty$ implies that $P(\lim \sup A_n) = 0$, and $\sum_{j=1}^{\infty} p_j' = \infty$ implies that $P(\lim \sup A_n) = 1$.

Cantelli proved that $\sum_{j=1}^{\infty} P(A_j) < \infty$ always implies that $P(\lim \sup A_n) = 0$.

Paul Lévy [4] clarified the general problem by proving the following theorem.

The subset K (or K') of the sample space Ω for which

$$\sum_{j=1}^{\infty} P(A_j|\alpha_1\alpha_2 \cdots \alpha_{j-1}) < \infty \text{ (or } = \infty)$$

and the subset H (or H') of Ω for which $\lim \sup A_n$ fails to occur (or occurs) differ at most by a set of probability 0. In other words $P(KH') = P(K'H) = 0$ and $P(KH) + P(K'H') = 1$. The hypothesis of the theorem proved in the next