NOTES

APPROXIMATION METHODS WHICH CONVERGE WITH PROBABILITY ONE[1, 2]

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1. Introduction and Summary. Let $H(y|x)$ be a family of distribution functions depending upon a real parameter x, and let $M(x) = \int_{-\infty}^{\infty} y \, dH(y \mid x)$ be the corresponding regression function. It is assumed $M(x)$ is unknown to the experimenter, who is, however, allowed to take observations on $H(y \mid x)$ for any value x.

Robbins and Monro [1] give a method for defining successively a sequence $\{x_n\}$ such that x_n converges to θ in probability, where θ is a root of the equation $M(x) = \alpha$ and α is a given number. Wolfowitz [2] generalizes these results, and Kiefer and Wolfowitz [3], solve a similar problem in the case when $M(x)$ has a maximum at $x = \theta$.

Using a lemma due to Loève [4], we show that in both cases x_n converges to θ with probability one, under weaker conditions than those imposed in [2] and [3]. Further we solve a similar problem in the case when $M(x)$ is the median of $H(y \mid x)$.

2. Approximation of the root of a regression equation. Let $M(x)$ be the regression function corresponding to the family $H(y \mid x)$. Assume $M(x)$ is a Lebesgue-measurable function satisfying:

A. $$| M(x) | \leqq c + d | x |, \qquad c, d \geqq 0;$$

B. $$\int_{-\infty}^{\infty} [y - M(x)]^2 \, dH(y \mid x) \leqq \sigma^2 < \infty;$$

C. $$M(x) < \alpha \text{ for } x < \theta, \qquad M(x) > \alpha \text{ for } x > \theta;$$

D. $$\inf_{\delta_1 \leqq |x-\theta| \leqq \delta_2} | M(x) - \alpha | > 0 \quad \text{for every pair of numbers} \quad (\delta_1, \delta_2)$$

with $0 < \delta_1 < \delta_2 < \infty$.

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[2] Kiefer and Wolfowitz had proved the main result of this paper in the Robbins-Monro case with bounded random variables. Their shorter proof for this case proceeds from the fact that a subsequence of $\{x_n\}$ converges to θ with probability one, and that $P\{\liminf x_n < c < d < \limsup x_n\} > 0$ for $\theta < c < d$ or $c < d < \theta$, yields an estimate of $\Sigma a_n d_n$ (see equation (9) of [J. Wolfowitz, *Ann. Math. Stat.*, Vol. 23 (1952), pp. 457–461]) which implies its divergence. This is a contradiction which proves the desired result.

Let $\{a_n\}$ be a sequence of positive numbers such that

$$(2.1) \qquad \text{(a) } \sum_{n=1}^{\infty} a_n = \infty, \qquad \text{(b) } \sum_{n=1}^{\infty} a_n^2 < \infty.$$

Let x_1 be an arbitrary number. Define a sequence of random variables recursively by

$$(2.2) \qquad x_{n+1} = x_n + a_n(\alpha - y_n)$$

where y_n is a random variable distributed according to $H(y \mid x_n)$. We use throughout a special case of Lemma 5.2 in [4]. We state it here as

LEMMA 1. *Let $\{v_n\}$ be a sequence of random variables such that $\sum_{n=1}^{\infty} Ev_n^2 < \infty$. Then $\sum_{j=1}^{n} [v_j - E(v_j \mid v_1, \cdots, v_{j-1})]$ converges to a random variable with probability one.*

LEMMA 2. *If* B *and* (2.1b) *hold, then the sequence $\{x_{n+1} - \sum_{j=1}^{n} a_j[\alpha - M(x_j)]\}$ converges to a random variable with probability one.*

PROOF. If we let $v_j = a_j[y_j - M(x_j)]$ then $E\{v_j^2\} = a_j^2E\{(y_j - M(x_j))^2\} \leqq a_j^2\sigma^2$. Hence $\sum_{n=1}^{\infty} E\{v_n^2\} < \infty$, by (2.1b), and Lemma 1 applies. Next we show

$$(2.3) \qquad E\{y_n - M(x_n) \mid y_1 - M(x_1), \cdots, y_{n-1} - M(x_{n-1})\} = 0.$$

To see this we note that given x_1 (a constant) we are given $M(x_1)$. But given $y_1 - M(x_1)$ and $M(x_1)$ we are given x_2, etc. But since $E\{y_n - M(x_n) \mid x_n\} = 0$, (2.3) follows. Thus we obtain that

$$(2.4) \qquad \sum_{j=1}^{n} a_j(y_j - M(x_j)) \quad \textit{converges with probability one.}$$

But this is clearly equivalent with the statement of the lemma, since $x_{n+1} = x_1 + \sum_{j=1}^{n} a_j(\alpha - y_j)$.

LEMMA 3. *If* A, B, C, *and* (2.1b) *hold, then x_n converges with probability one.*

PROOF. To begin with we establish

$$(2.5) \qquad P\{\lim_{n\to\infty} x_n = +\infty\} + P\{\lim_{n\to\infty} x_n = -\infty\} = 0.$$

For suppose $\{x_n\}$ is a sample sequence with $\lim_{n\to\infty} x_n = +\infty$. Then we have $x_n \leqq \theta$ for only finitely many n. Hence for n sufficiently large we have $a_n(\alpha - M(x_n)) < 0$ from C. But then $\lim_{n\to\infty} [x_{n+1} - \sum_{j=1}^{n} a_j(\alpha - M(x_j))] = +\infty$. But this can only happen with probability zero, from Lemma 2, establishing (2.5). Now suppose the conclusion of the lemma is false. Then by virtue of Lemma 2 and (2.5) there exists a set of sample sequences of positive probability with the following properties:

$$(2.6) \qquad \begin{cases} \text{(a)} \quad x_{n+1} - \sum_{j=1}^{n} a_j(\alpha - M(x_j)) \textit{ converges to a finite number.} \\ \text{(b)} \quad \liminf x_n < \limsup x_n \end{cases}$$

for every sample sequence in the set. Let $\{x_n\}$ be such a sequence and assume $\limsup x_n > \theta$. (A similar argument handles the situation $\limsup x_n \leqq \theta$.) Then we may choose numbers a and b satisfying

$$a > \theta, \qquad \liminf x_n < a < b < \limsup x_n. \tag{2.7}$$

Since $\lim_{n\to\infty} a_n = 0$ from (2.1b) and because of (2.6) we may choose N so large that $N \leqq n < m$ implies

$$\begin{cases} \text{(a)} \quad a_n \leqq \min\left\{\dfrac{1}{3d}, \dfrac{b-a}{3[\,|\alpha| + c + d\,|\theta|\,]}\right\} \\ \text{(b)} \quad \left| x_m - x_n - \sum_{j=n}^{m-1} a_j(\alpha - M(x_j)) \right| \leqq \dfrac{b-a}{3}. \end{cases} \tag{2.8}$$

Now choose m and n such that

$$\begin{cases} \text{(a)} & N \leqq n < m \\ \text{(b)} & x_n < a, \qquad x_m > b \\ \text{(c)} & n < j < m \text{ implies } a \leqq x_j \leqq b. \end{cases} \tag{2.9}$$

This can clearly be done. Then we obtain

$$x_m - x_n \leqq \frac{(b-a)}{3} + \sum_{j=n}^{m-1} a_j(\alpha - M(x_j)) \leqq \frac{(b-a)}{3} + a_n(\alpha - M(x_n)) \tag{2.10}$$

since for $n < j < m$, (2.9) and C imply $a_j(\alpha - M(x_j)) < 0$. If $\theta < x_n$, we obtain

$$x_m - x_n \leqq (b-a)/3 \tag{2.11}$$

which is a contradiction to (2.9). Suppose then that $\theta \geqq x_n$. Applying A we have

$$\begin{aligned} |M(x_n)| \leqq c + d\,|x_n| &\leqq c + d\,|\theta| + d\,|\theta - x_n| \\ &\leqq c + d\,|\theta| + d(x_m - x_n). \end{aligned} \tag{2.12}$$

Hence by applying (2.10) we have

$$x_m - x_n \leqq (b-a)/3 + a_n[\,|\alpha| + c + d\,|\theta|\,] + a_n d(x_m - x_n). \tag{2.13}$$

Thus $x_m - x_n \leqq 2(b-a)/3(1 - a_n d) \leqq b - a$ by (2.8). But this is again a contradiction to (2.9), proving the lemma.

Theorem 1. *If conditions* A *through* (2.1b) *hold, then x_n converges to θ with probability one.*

Proof. Suppose $P\{\lim_n x_n = x\} = 1$, as guaranteed by Lemma 3, and suppose further

$$P\{x \neq \theta\} > 0. \tag{2.14}$$

Then we may choose ϵ_1 and ϵ_2 with, say, $\theta < \epsilon_1 < \epsilon_2 < \infty$ such that

$$P\{\epsilon_1 < x < \epsilon_2\} > 0. \tag{2.15}$$

(Otherwise we may choose ϵ_1 and ϵ_2 with $-\infty < \epsilon_1 < \epsilon_2 < \theta$). Then for every sample sequence $\{x_n\}$ for which $\lim_n x_n = x$, with $\epsilon_1 < x < \epsilon_2$, we have

$$\epsilon_1 \leqq x_n \leqq \epsilon_2 \tag{2.16}$$

for all n sufficiently large so that the set of sample sequences $\{x_n\}$ satisfying (2.16) has positive probability. But from Lemma 2 and Lemma 3, we have for almost all such sequences that

$$\sum_{j=1}^{n} a_j(\alpha - M(x_j)) \quad \text{converges.} \tag{2.17}$$

But this is contradicted by D and (2.1a), proving the theorem.

3. Approximation of the maximum of a regression function. Let $M(x)$ again be a measurable regression function satisfying B and also

E. $M(x)$ is strictly increasing for $x < \theta$, and strictly decreasing for $x > \theta$.

F. There exist positive numbers ρ and R such that $|x' - x''| < \rho$ implies $|M(x') - M(x'')| < R$.

G. For every $\delta > 0$ there exists a positive number $\pi(\delta)$ such that $|x - \theta| > \delta$ implies

$$\inf_{\delta/2 > \epsilon > 0} \frac{|M(x + \epsilon) - M(x - \epsilon)|}{\epsilon} > \pi(\delta).$$

Let $\{a_n\}$ and $\{c_n\}$ be sequences of positive numbers satisfying

H. (i) $c_n \to 0$; (ii) $\sum_{n=1}^{\infty} a_n = \infty$; (iii) $\sum_{n=1}^{\infty} (a_n/c_n)^2 < \infty$.

We define a recursive scheme as follows. Let x_1 be an arbitrary number. Define

$$x_{n+1} = x_n + (a_n/c_n)(y_{2n} - y_{2n-1})$$

where y_{2n} and y_{2n-1} are independent random variables distributed according to $H(y \mid x_n + c_n)$ and $H(y \mid x_n - c_n)$ respectively. Then we have

THEOREM 2. *If conditions* B *and* E *through* H *hold, then* $P\{\lim x_n = \theta\} = 1$.

The proof of the theorem will be omitted here. It consists in repeating the proofs of Lemma 2, Lemma 3, and Theorem 1, with obvious modifications.

We note that conditions B and E through H represent a weakening of the conditions imposed in [3], since conditions (2.5) and (2.8) of that paper are not used here.

4. Estimation of the value at which a conditional median vanishes.

Suppose $H(y \mid x)$ is a family of distribution functions such that, for a given number α, $H(\alpha \mid x)$ is a measurable function of x. Assume $M(x)$ is the (not necessarily unique) median of $H(y \mid x)$. We assume the following conditions on $M(x)$ and $H(\alpha \mid x)$:

$$M(x) < \alpha \text{ for } x < \theta, \qquad M(x) > \alpha \text{ for } x > \theta. \tag{4.1}$$

By this we mean that if x is less than θ, then every median of $H(y \mid x)$ is less than α, and similarly if x is greater than θ.

$$\inf_{\delta_1 \leq |x-\theta| \leq \delta_2} |H(\alpha \mid x) - \tfrac{1}{2}| > 0 \tag{4.2}$$

for every pair of numbers (δ_1, δ_2) with $0 < \delta_1 < \delta_2 < \infty$. Let $\{a_n\}$ be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} a_n = \infty, \qquad \sum_{n=1}^{\infty} a_n^2 < \infty. \tag{4.3}$$

Define a recursive approximation scheme as follows. Let x_1 be arbitrary and define

$$x_{n+1} = x_n + a_n z_n \tag{4.4}$$

where $z_n = +1$ if $y_n \leqq \alpha$ and $z_n = -1$ if $y_n > \alpha$, and y_n is a random variable distributed according to $H(y \mid x_n)$. Then, by applying Theorem 1 with $\alpha = 0$ and $y_n = -z_n$, we obtain

THEOREM 3. *If conditions* (4.1), (4.2), *and* (4.3) *hold, then* $P\{\lim x_n = \theta\} = 1$.

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REFERENCES

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A NOTE ON THE ROBBINS-MONRO STOCHASTIC APPROXIMATION METHOD[1]

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Introduction. The almost certain convergence of the RM process and related stochastic approximation procedures is proved by Blum [1] in a paper appearing elsewhere in this issue. In the present note we consider the method originally proposed by Robbins and Monro [2] with a further restriction on the constants a_n. Our aim is to obtain, by elementary methods, an estimate of the order of magnitude of $b_n = E(x_n - \theta)^2$. This estimate is sharp enough to enable us to prove strong convergence for certain types of sequences a_n. The method adopted in [1], while being more general, does not yield information about the behavior

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