# TRUNCATED LIFE TESTS IN THE EXPONENTIAL CASE

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**1. Introduction and Summary.** It is frequently desirable on practical grounds to terminate a life test by a preassigned time $T_0$. In this paper we consider life tests which are truncated as follows. With $n$ items placed on test, it is decided in advance that the experiment will be terminated at min $(X_{r_0,n}, T_0)$, where $X_{r_0,n}$ is a random variable equal to the time at which the $r_0$th failure occurs and $T_0$ is a truncation time, beyond which the experiment will not be run. Both $r_0$ and $T_0$ are assigned before experimentation starts. If the experiment is terminated at $X_{r_0,n}$ (that is, if $r_0$ failures occur before time $T_0$), then the action in terms of hypothesis testing is the rejection of some specified null-hypothesis. If the experiment is terminated at time $T_0$ (that is, if the $r_0$th failure does not occur before time $T_0$), then the action in terms of hypothesis testing is the acceptance of some specified null-hypothesis.

While truncated procedures can be considered for any life distribution, we limit ourselves here to the case where the underlying life distribution is specified by a p.d.f. of the exponential form, $f(x; \theta) = \theta^{-1}e^{-x/\theta}$, $x > 0$, $\theta > 0$. The practical justification for using this kind of distribution as a first approximation to a number of test situations is discussed in a recent paper by Davis [1]. It is a common assumption for electron tube life.

Two situations are considered. The first is the nonreplacement case in which a failure occurring during the test is not replaced by a new item. The second is the replacement case where failed items are replaced at once by new items drawn at random from the same p.d.f. as the original $n$ items. Formulae are given for $E_\theta(r)$, the expected number of observations to reach a decision; for $E_\theta(T)$, the expected waiting time to reach a decision; and for $L(\theta)$, the probability of accepting the hypothesis that $\theta = \theta_0$, the value associated with the null-hypothesis, when $\theta$ is the true value. Some procedures are worked out for finding truncated tests meeting specified conditions, and practical illustrations are given.

It is an intrinsic feature of all life test decision procedures that they are in some sense truncated, although not necessarily by a fixed time $T_0$. In Section 3 we give exact formulae for $E_\theta(r)$ and $E_\theta(T)$ for a decision procedure given in [2]. There is a close relation between these results and those in Section 2.

**2. Derivation of a truncated test in the nonreplacement and replacement case.** It will be assumed throughout this section that the underlying p.d.f. of the life of items is given by $f(x; \theta) = \theta^{-1}e^{-x/\theta}$, $x > 0$, $\theta > 0$. In the nonreplacement case, $n$ items are drawn at random from the population and placed on life test. Items which fail are not replaced and the experiment is truncated at time min $(X_{r_0,n}$,

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$T_0$), where $X_{r_0,n}$ is the time when the $r_0$th failure occurs and $r_0$ and $T_0$ will be taken as preassigned. $T_0$ is a truncation time beyond which the experiment does not run. The variate is considered to be time for convenience only. It is perfectly clear that it can be other things, depending on the physical applications one is concerned with. Generally the variate will be nonnegative.

Since the probability of an item failing before time $T_0$ is given by $p_\theta = 1 - e^{-T_0/\theta}$, it follows from the binomial law that the probability of reaching a decision requiring exactly $k$ failures is

$$\text{(1)} \quad \Pr(r = k \mid \theta) = b(k; n, p_\theta) = \binom{n}{k} p_\theta^k (1 - p_\theta)^{n-k}, \quad k = 0, 1, 2, \cdots, r_0 - 1$$

$$\text{(2)} \quad \Pr(r = r_0 \mid \theta) = 1 - \sum_{k=0}^{r_0-1} b(k; n, p_\theta).$$

The expected number of observations to reach a decision is

$$\text{(3)} \quad E_\theta(r) = \sum_{k=0}^{r_0} k \Pr(r = k \mid \theta).$$

It can be readily shown that (3) simplifies to

$$\text{(4)} \quad E_\theta(r) = np_\theta \left[ \sum_{k=0}^{r_0-2} b(k; n - 1, p_\theta) \right] + r_0 \left[ 1 - \sum_{k=0}^{r_0-1} b(k; n, p_\theta) \right].$$

This is in a convenient form for calculation. For any preassigned $n$, $T_0$, and $r_0$, $E_\theta(r)$ can be found easily from the Binomial Tables [8] or the Tables of the Incomplete Beta Function [6].

We now wish to prove that $E_\theta(T)$, the expected waiting time to reach a decision based on the stopping rule min $(X_{r_0,n}, T_0)$, is

$$\text{(5)} \quad E_\theta(T) = \sum_{k=1}^{r_0} \Pr(r = k \mid \theta) E_\theta(X_{k,n}),$$

where $E_\theta(X_{k,n})$ is the unconditional expected waiting time (measured from $t = 0$) to observe the $k$th failure in the random sample of size $n$ drawn from the underlying exponential p.d.f.

To prove (5), we note first that $E_\theta(T)$ is

$$\text{(6)} \quad E_\theta(T) = T_0 \left[ \sum_{k=0}^{r_0-1} b(k; n, p_\theta) \right] + \sum_{k=r_0}^{n} b(k; n, p_\theta) E_\theta(X_{r_0,n} \mid r = k).$$

Furthermore $E_\theta(X_{r_0,n})$, the unconditional expected waiting time to get the $r_0$th failure, is

$$\text{(7)} \quad \begin{aligned} E_\theta(X_{r_0,n}) = \sum_{k=0}^{r_0-1} b(k; n, p_\theta) E_\theta(X_{r_0,n} \mid r = k) \\ + \sum_{k=r_0}^{n} b(k; n, p_\theta) E_\theta(X_{r_0,n} \mid r = k). \end{aligned}$$

From (6) and (7) we get

$$E_\theta(T) = E_\theta(X_{r_0,n}) + \sum_{k=0}^{r_0-1} b(k; n, p_\theta)[T_0 - E_\theta(X_{r_0,n} \mid r = k)]. \tag{8}$$

Since the underlying distribution is exponential, it can be verified in the nonreplacement case from results in [3] that

$$E_\theta(X_{r_0,n} \mid r = k) = T_0 + E_\theta(X_{r_0-k,n-k}), \qquad k = 1, 2, \cdots, r_0 - 1 \tag{9}$$

where $E_\theta(X_{r_0-k,n-k})$ is the unconditional expected waiting time to get the $(r_0 - k)$th failure in a random sample of size $(n - k)$. It has been shown in [2] that for $1 \leqq k \leqq n$,

$$E_\theta(X_{k,n}) = \theta\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-k+1}\right). \tag{10}$$

Therefore

$$E_\theta(X_{r_0-k,n-k}) = E_\theta(X_{r_0,n}) - E_\theta(X_{k,n}), \qquad 1 \leqq k \leqq r_0. \tag{11}$$

Using (9) and (11) in (8) gives the desired formula (5).

In the replacement case the test is started with $n$ items and any item that fails is replaced at once by a new item drawn at random from the underlying p.d.f.; thus the number of items under test is always $n$. As in the nonreplacement test, case experimentation is truncated at time min $(X_{r_0,n}, T_0)$, where $X_{r_0,n}$ is the time (measured from the beginning of the entire experiment) when the $r_0$th failure occurs, and $T_0$ is a preassigned truncation time.

Since the underlying distribution is exponential with mean life $\theta$, the replacement of failed items by new items makes the life test a Poisson process with occurrence rate $\lambda_\theta = n/\theta$.

Thus the probability of reaching a decision requiring exactly $k$ failures is

$$\Pr(r = k \mid \theta) = p(k; \lambda_\theta T_0) = \frac{1}{k!} e^{-nT_0/\theta}(nT_0/\theta)^k, \quad k = 0, 1, 2, \cdots, r_0 - 1 \tag{12}$$

$$\Pr(r = r_0 \mid \theta) = 1 - \sum_{k=0}^{r_0-1} p(k; \lambda_\theta T_0). \tag{13}$$

The expected number of observations to reach a decision is

$$E_\theta(r) = \sum_{k=0}^{r_0} k \Pr(r = k \mid \theta). \tag{14}$$

It can readily be shown that (14) simplifies to

$$E_\theta(r) = \lambda_\theta T_0 \left[\sum_{k=0}^{r_0-2} p(k; \lambda_\theta T_0)\right] + r_0 \left[1 - \sum_{k=0}^{r_0-1} p(k; \lambda_\theta T_0)\right]. \tag{15}$$

This is in a convenient form for calculation. For any preassigned $n$, $T_0$ and $r_0$, $E_\theta(r)$ can be found easily from Molina's tables of the Poisson distribution [5] or from the tables on the incomplete $\Gamma$-distribution [7].

The expected waiting time to reach a decision is given by a particularly simple formula in the replacement case. It is

$$(16) \qquad E_\theta(T) = (\theta/n)E_\theta(r).$$

The proof of (16) is analogous to the proof of (5) in the nonreplacement case. Thus analogous to (8) we have

$$(17) \qquad E_\theta(T) = E_\theta(X_{r_0,n}) + \sum_{k=0}^{r_0-1} p(k; \lambda_\theta T_0)[T_0 - E_\theta(X_{r_0,n} \mid r = k)].$$

Analogous to (9) we have

$$(18) \qquad E_\theta(X_{r_0,n} \mid r = k) = T_0 + E_\theta(X_{r_0-k,n}), \qquad k = 1, 2, \cdots, r_0 - 1.$$

Furthermore

$$(19) \qquad E_\theta(X_{r_0-k,n}) = E_\theta(X_{r_0,n}) - E_\theta(X_{k,n}) = (r_0 - k)\theta/n, \qquad 1 \leqq k \leqq r_0,$$

since the unconditional expected waiting time to get exactly $s$ failures in a replacement situation is $E_\theta(X_{s,n}) = s\theta/n$, for any integer $s$. Substituting (18) and (19) in (17) yields (16).

It is interesting to note that in analogy with (5) in the nonreplacement case we can write (16) as

$$(20) \qquad E_\theta(T) = \sum_{k=1}^{r_0} \Pr(r = k \mid \theta)E_\theta(X_{k,n}).$$

The unconditional waiting times $E_\theta(X_{k,n})$ are given by (10) in the nonreplacement case, by $k\theta/n$ in the replacement case.

Suppose the truncation rule is such that the hypothesis $H_0: \theta = \theta_0$ is accepted if $\min(X_{r_0,n}, T_0) = T_0$, that is, if the waiting time required to observe $X_{r_0,n}$ is more than $T_0$. Then if $L(\theta)$ is defined as the probability of accepting $\theta = \theta_0$ when $\theta$ is true, it follows in either the replacement or the nonreplacement case that

$$(21) \qquad L(\theta) = 1 - \Pr(r = r_0 \mid \theta),$$

where $\Pr(r = r_0 \mid \theta)$ is given by (2) in the nonreplacement case and by (13) in the replacement case.

**3. A test based on the first $r$ out of $n$ ordered observations.** In [2] it was proved, in the nonreplacement case, that when testing the hypothesis $H_0: \theta = \theta_0$ against any simple alternative $\theta = \theta_1$ ($\theta_1 < \theta_0$), the "best" region of acceptance for $H_0$ (in the sense of Neyman and Pearson), based on the first $r$ out of $n$ ordered observations from an exponential distribution, is of the form $\hat{\theta}_{r,n} > C$, where

$$(22) \qquad \hat{\theta}_{r,n} = \left[\frac{1}{r}\sum_{i=1}^{r} x_{i,n} + (n - r)x_{r,n}\right]$$

both $r$ and $n$ being preassigned integers.

One could interpret the decision rule $\hat{\theta}_{r,n} > C$ to mean that we wait until time $x_{r,n}$ then compute $\hat{\theta}_{r,n}$ and make the appropriate decision. However, this procedure clearly wastes information since we are able to observe the life test continuously. We will now show that, if continuous observation is taken into account, often we can shorten the waiting time to reach a decision and reduce the number of items failed. More precisely, suppose that at some moment $t$ there are exactly $k$ failures, $0 \leqq k \leqq r - 1$, and that the observed total life $V(t)$,

$$V(t) = \sum_{i=1}^{k} x_{i,n} + (n - k)t \tag{23}$$

is greater than $rC$. The $k$ items which fail by time $t$ contribute $\sum_{i=1}^{k} x_{i,n}$ to $V(t)$. The $(n - k)$ items which have not failed contribute the amount $(n - k)t$. In particular if $t = x_{r,n}$, then $V(x_{r,n}) = \sum_{i=1}^{r} x_{i,n} + (n - r)x_{r,n} = r\hat{\theta}_{r,n}$. Since $V(t)$ is monotonically increasing in $t$, we know that $V(x_{r,n}) \geqq V(t) > rC$, and thus we should stop experimentation at time $t$ and accept $H_0$. More generally a decision rule having precisely the same O.C. curve as $\hat{\theta}_{r,n} > C$, but requiring on the average fewer failures and a shorter decision time, is as follows:

(a) Continue experimentation so long as $V(t) < rC$ and $0 \leqq k \leqq r - 1$.

(b) Stop experimentation with acceptance of $H_0$ as soon as $V(t) > rC$ and $0 \leqq k \leqq r - 1$.

(c) Stop experimentation at $x_{r,n}$ with rejection of $H_0$ if $V(t) < rC$ for all $t \leqq x_{r,n}$.

Note: This means that acceptance of $H_0$ takes place between failure times, and always before time $x_{r,n}$.

We now proceed to find certain useful properties of the truncated rule based on $V(t)$. To find these properties, we first remark (defining $x_{0,n}$ as zero) that

$$\sum_{i=1}^{r} x_{i,n} + (n - r)x_{r,n} \equiv \sum_{i=1}^{r} (n - i + 1)(x_{i,n} - x_{i-1,n}). \tag{24}$$

Introducing new random variables defined by

$$\xi_1 = nx_{1,n}, \qquad \xi_i = (n - i + 1)(x_{i,n} - x_{i-1,n}), \qquad i = 1, 2, \cdots, r \tag{25}$$

$\hat{\theta}_{r,n} > C$ can be written as

$$\sum_{i=1}^{r} \xi_i > rC. \tag{26}$$

The $\xi_i$ are mutually independent random variables, each distributed with common p.d.f. $\theta^{-1}e^{-x/\theta}$, $x > 0$, $\theta > 0$. If we interpret $\xi_i$ as the time interval between the $(i - 1)$st and $i$th event in a Poisson process having mean occurrence rate $\lambda = \theta^{-1}$, it is clear that $\sum_{i=1}^{r} \xi_i > rC$, if and only if $k$, the number of events in an interval of length $rC$, is $0 \leqq k \leqq r - 1$. If the number of events in such an interval is $\geqq r$, $\sum_{i=1}^{r} \xi_i \leqq rC$. Thus the probability of reaching a decision requiring exactly $\rho = k$ failures is

$$\Pr(\rho = k \mid \theta) = p(k; \mu_\theta), \qquad k = 0, 1, 2, \cdots, r_0 - 1, \tag{27}$$

$$\Pr(\rho = r \mid \theta) = 1 - \sum_{k=0}^{r-1} p(k; \mu_\theta). \tag{28}$$

In (27) and (28), $\mu_\theta = rC/\theta$ and $p(k; \mu_\theta) = \mu_\theta^k e^{-\mu_\theta}/k!$. The expected number of observations to reach a decision is

$$(29) \quad E_\theta(\rho) = \sum_{k=0}^{r} k \Pr(\rho = k \mid \theta) = \mu_\theta \left[ \sum_{k=0}^{r-2} p(k; \mu_\theta) \right] + r \left[ 1 - \sum_{k=0}^{r-1} p(k; \mu_\theta) \right].$$

It can be verified that $E_\theta(T)$ for the $V(t)$ procedure can be written (as in the replacement or nonreplacement case) as

$$(30) \qquad E_\theta(T) = \sum_{k=1}^{r} \Pr(\rho = k \mid \theta) E_\theta(X_{k,n}),$$

where $\Pr(\rho = k \mid \theta)$ is given by (27) and (28) and $E_\theta(X_{k,n})$ is given by (10). Finally $L(\theta)$, the probability of accepting $\theta = \theta_0$ when $\theta$ is true, is given by $L(\theta) = \sum_{k=0}^{r-1} p(k; \mu_\theta)$.

Up to this point in the present section we have been treating the nonreplacement situation. It is interesting to see what happens if failed items are replaced at once by new items drawn from the p.d.f. $\theta^{-1}e^{-x/\theta}$. As in Section 2, let $x_{k,n}$ be the time when the $k$th failure occurs (whether it be an original item or replacement item) measured from the beginning of the experiment. It can be shown, in the replacement case, that if one starts with $n$ items, then the "best" region of acceptance, in the Neyman-Pearson sense, for testing a hypothesis $H_0$ that $\theta = \theta_0$ against alternatives of the form $\theta = \theta_1(\theta_1 > \theta_0)$, based on the first $r$ failure times $x_{1,n}, x_{2,n}, \cdots, x_{r,n}$, is of the form $\hat{\theta}_{r,n} > C$, where $\hat{\theta}_{r,n}$ is now simply equal to

$$(31) \qquad \hat{\theta}_{r,n} = n x_{r,n}/r.$$

It follows that the region of acceptance for $H_0$ is of the form $x_{r,n} > C^* = rC/n$. Use of $x_{r,n} > C^*$ as a region of acceptance means in words that the test is terminated at min $(X_{r,n}, C^*)$ with acceptance of $H_0$ if truncation occurs at $C^*$ and rejection of $H_0$ if truncation occurs at $X_{r,n}$. This is precisely the test treated in Section 2 with $r = r_0$ and $C^* = T_0$.

**4. Some computational remarks.** In Section 2 we gave formulae for the O.C. curve, the expected waiting time, and expected number of items failed in the course of reaching a decision for any preassigned $n$, $T_0$, and $r_0$. We now give a procedure for finding the appropriate truncated test (that is, for finding $r_0$ and $n$) when the truncation time $T_0$ is preassigned and the O.C. curve is required (for preassigned type I error, $\alpha$, and type II error, $\beta$) to be such that $L(\theta_0) \geqq 1 - \alpha$ and $L(\theta_1) \leqq \beta$. Here $\theta_0$ and $\theta_1$ are preassigned with $\theta_0 > \theta_1$.

To find such a test we recall [2] that the best acceptance region of size $\alpha$ for the hypothesis $\theta = \theta_0$ (against any alternative $\theta_1 < \theta_0$), based on the first $r$ out of $n$ failures, for preassigned $r$ and $n$, is

$$(32) \qquad \hat{\theta}_{r,n} > C = \theta_0 \chi^2_{1-\alpha}(2r)/2r.$$

(A chi square variable with $n$ degrees of freedom is denoted as $\chi^2(n)$. The constant $\chi^2_\gamma(n)$ is defined by the equality $\Pr(\chi^2(n) > \chi^2_\gamma(n)) = \gamma$.) In order that the

TABLE 1

*Values of r (upper numbers) and of $\chi^2_{1-\alpha}(2r)/2$. (lower numbers) such that the test based on using $\hat{\theta}_{r,n} > C = \theta_0\chi^2_{1-\alpha}(2r)/2r$ as acceptance region for $\theta = \theta_0$ will have $L(\theta_0) = 1 - \alpha$ and $L(\theta_1) \geqq \beta$*

| $\theta_0/\theta_1$ | $\alpha = .01$ | | | $\alpha = .05$ | | | $\alpha = .10$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $\beta = .01$ | $\beta = .05$ | $\beta = .10$ | $\beta = .01$ | $\beta = .05$ | $\beta = .10$ | $\beta = .01$ | $\beta = .05$ | $\beta = .01$ |
| 3/2 | 136 | 101 | 83 | 95 | 67 | 55 | 77 | 52 | 41 |
| | 110.4 | 79.1 | 63.3 | 79.6 | 54.1 | 43.4 | 66.0 | 43.0 | 33.0 |
| 2 | 46 | 35 | 30 | 33 | 23 | 19 | 26 | 18 | 15 |
| | 31.7 | 22.7 | 18.7 | 24.2 | 15.7 | 12.4 | 19.7 | 12.8 | 10.3 |
| 5/2 | 27 | 21 | 18 | 19 | 14 | 11 | 15 | 11 | 9 |
| | 16.4 | 11.8 | 9.62 | 12.4 | 8.46 | 6.17 | 10.3 | 7.02 | 5.43 |
| 3 | 19 | 15 | 13 | 13 | 10 | 8 | 11 | 8 | 6 |
| | 10.3 | 7.48 | 6.10 | 7.69 | 5.43 | 3.98 | 7.02 | 4.66 | 3.15 |
| 4 | 12 | 10 | 9 | 9 | 7 | 6 | 7 | 5 | 4 |
| | 5.43 | 4.13 | 3.51 | 4.70 | 3.29 | 2.61 | 3.90 | 2.43 | 1.75 |
| 5 | 9 | 8 | 7 | 7 | 5 | 4 | 5 | 4 | 3 |
| | 3.51 | 2.91 | 2.33 | 3.29 | 1.97 | 1.37 | 2.43 | 1.75 | 1.10 |
| 10 | 5 | 4 | 4 | 4 | 3 | 3 | 3 | 2 | 2 |
| | 1.28 | .823 | .823 | 1.37 | .818 | .818 | 1.10 | .532 | .532 |

test have an O.C. curve for which $L(\theta_0) = 1 - \alpha$ and $L(\theta_1) \leqq \beta$, we need to choose $r$ suitably. The appropriate values of $r$ for certain values of $\alpha$, $\beta$, and $\theta_0/\theta_1$ are given in Table 1. For values of $\alpha$, $\beta$, and $\theta_0/\theta_1$ not given in the table, the appropriate $r$ to use is the smallest integer $r$ such that $\chi^2_{1-\alpha}(2r)/\chi^2_{\beta}(2r) \geqq \theta_1/\theta_0$.

It is now an easy matter, in the replacement case, to find a truncated test meeting the conditions prescribed in the opening paragraph of this section. In view of the last two paragraphs of Section 3, the appropriate $r_0$ in the replacement case is given by the values in Table 1. Furthermore, we want $T_0 = C^* = rC/n = \theta_0\chi^2_{1-\alpha}(2r)/2n$. Since $n$ must be an integer, the equality can be satisfied only approximately. For all practical purposes $n$ can be chosen as

$$n = [\theta_0\, \chi^2_{1-\alpha}(2r_0)/2T_0] \tag{33}$$

where $[x]$ means the greatest integer $\leqq x$. It is interesting to note that the appropriate $n$, for fixed $\alpha$ and $\beta$, is inversely proportional to the time of truncation $T_0$. Thus, for example, to reduce the truncation time by a factor of two requires doubling $n$. It is clear from (33) that the values of $\chi^2_{1-\alpha}(2r_0)/2$ are useful to tabulate. These are given below the associated $r_0$ in Table 1.

The O.C. curve of the test min $[X_{r_0,n}, T_0]$, where $r_0$ is given by Table 1 and $n$ by (33), is such that $L(\theta_0) \geqq 1 - \alpha$, but in some cases $L(\theta_1)$ may be slightly $>$ $\beta$. This can be avoided in either of two ways. One way is to give the experimenter the freedom to use, instead of $T_0$, the slightly larger truncation time $T_0' = \theta_0\chi^2_{1-\alpha}(2r)/n$; the test based on min $[X_{r_0,n}, T_0']$ will have $L(\theta_0) = 1 - \alpha$ and $L(\theta_1) \leqq \beta$. The other way is to use $n + 1$ items throughout the test, and to use, instead of $T_0$, the slightly smaller truncation time $T_0'' = \theta_0\chi^2_{1-\alpha}(2r_0)/(n + 1)$; The test based on min $[X_{r_0,n+1}, T_0'']$ will have $L(\theta_0) = 1 - \alpha$ and $L(\theta_1) \leqq \beta$. In most cases it will be a matter of indifference which procedure one adopts.

The most direct (and also the most lengthy) procedure for finding a truncated nonreplacement test meeting the conditions prescribed in the opening paragraph of this section is to note that such a test is equivalent to a binomial situation in which we test $p_0 = 1 - e^{-T_0/\theta_0}$ against $p_1 = 1 - e^{-T_0/\theta_1}$, and want the O.C. curve to be such that $L(p_0) \geqq 1 - \alpha$ and $L(p_1) \leqq \beta$. In binomial terms, we are seeking a sample size $n$ and a rejection number $r_0$ such that we will accept the hypothesis that $p = p_0$ if the number of defectives (failures) in the sample $\leqq r_0 - 1$. The hypothesis that $p = p_0$ will be rejected if the number of defectives in a sample of size $n$ is $\geqq r_0$. The detailed calculations can be carried out in any given situation by using the Binomial Tables [8] or Tables of the Incomplete Beta Function [6].

While the procedure described in the preceding paragraph can always be worked out, it is both tedious and time consuming. If the values of $\alpha$ and $\beta$ are small and the ratio $\theta_0/T_0$ is substantially more than one (say $\geqq 3$), then the required $n$ will be fairly large. In such cases a somewhat less exact, but much shorter, way of finding the appropriate $r_0$ and $n$ can be used. As $r_0$ use the same value as in the replacement case. Let the sample size $n = [r_0/(1 - e^{-T_0/C})]$, where $C = \theta_0\chi^2_{1-\alpha}(2r_0)/2r_0$. The justification for this approximation is briefly the following. If $n$ is substantially more than $r_0$, then the O.C. curve based on the rule $\beta_{r_0,n}x_{r_0,n} > C$, where $\beta_{r_0,n} = 1/E(X_{r_0,n})$ is very close to the O.C. curve based on the rule $\hat{\theta}_{r_0,n} > C$. To truncate experimentation at time $T_0$ means finding an $n$ such that $C/\beta_{r_0,n} = T_0$. When $n$ is large $1/\beta_{r_0,n} \sim \log [n/(n - r)]$. After some simple manipulation we arrive at the above formula for $n$.

In Table 2, we give some values of $n$ computed by this formula for $\alpha = .01, .05$; $\beta = .01, .05$; $\theta_0/\theta_1 = 2, 3, 5$; and $\theta_0/T_0 = 3, 5, 10, 20$. These values have been checked by computing $L(\theta_0)$ and $L(\theta_1)$; the O.C. curve does come very close to meeting the requirements $L(\theta_0) \geqq 1 - \alpha$ and $L(\theta_1) \leqq \beta$.

TABLE 2

*Values of n to be used in truncated nonreplacement procedures.*

| $\alpha, \beta$ | .01, .01 | | | .01, .05 | | | .05, .01 | | | .05, .05 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta_0/\theta_1$ | 2 | 3 | 5 | 2 | 3 | 5 | 2 | 3 | 5 | 2 | 3 | 5 |
| $\theta_0/T_0$ | | | | | | | | | | | | |
| 3 | 120 | 41 | 15 | 87 | 30 | 13 | 90 | 30 | 13 | 59 | 21 | 8 |
| 5 | 182 | 61 | 22 | 132 | 45 | 18 | 138 | 45 | 20 | 90 | 32 | 12 |
| 10 | 340 | 113 | 39 | 245 | 82 | 33 | 258 | 83 | 36 | 169 | 59 | 22 |
| 20 | 657 | 216 | 74 | 472 | 157 | 62 | 499 | 160 | 69 | 325 | 113 | 42 |

## 5. Examples.

PROBLEM 1. Find a truncated replacement plan for which $T_0 = 500$ hours, which will accept a lot with mean life $= 10{,}000$ hours at least 95 per cent of the time and reject a lot with mean life $= 2{,}000$ hours at least 95 per cent of the time. Compute $L(\theta)$, $E_\theta(T)$, and $E_\theta(r)$ at $\theta = 10{,}000$ and $\theta = 2{,}000$, respectively.

SOLUTION. In this case $\theta_0 = 10{,}000$, $\theta_1 = 2{,}000$, $\alpha = \beta = .05$. Since $\theta_0/\theta_1 = 5$, it follows from Table 1 that $r_0 = 5$. Since $\theta_0/T_0 = 20$, $n = [(1.97)(20)] = 39$. Thus the following truncated replacement plan meets the requirements. Start the life test with $n = 39$ items. As soon as an item fails, replace it by a new item. Accept the lot if $\min [X_{5,39}, 500] = 500$ and reject the lot if $\min [X_{5,39}, 500] = X_{5,39}$. If the lot is rejected, experimentation is stopped at $X_{5,39}$, the time of occurrence of the fifth failure.

For $\theta = 10{,}000$, $\lambda_\theta T_0 = 1.95$. Using the tables [5] and (21), it is easily verified that $L(\theta) = .952$. Substituting in (15) and (16) respectively gives $E_\theta(r) = 1.93$ and $E_\theta(T) = 495$. For $\theta = 2{,}000$, $\lambda_\theta T_\theta = 9.75$. For this value of $\theta$, $L(\theta) = .034$, $E_\theta(r) = 4.95$, and $E_\theta(T) = 254$.

PROBLEM 2. Same as 1 except that we want a nonreplacement procedure.

SOLUTION. $r_0 = 5$. According to Table 2, the sample size is $n = 42$. For $\theta = 10{,}000$, $T_0/\theta = .05$, and $p_\theta = 1 - e^{-T_0/\theta} = .049$. Using the table [8], one finds $L(\theta) = .946$. Substituting in (4) and (5) respectively gives $E_\theta(r) = 2.02$ and $E_\theta(T) = 494$. For $\theta = 2{,}000$, $T_0/\theta = .25$. For this value of $\theta$, $L(\theta) = .031$, $E_\theta(r) = 4.91$, and $E_\theta(T) = 248$.

PROBLEM 3. Consider the truncated replacement plan meeting the conditions of Problem 1. For what values of $\theta$ is $L(\theta) = .5$? What are $E_\theta(r)$ and $E_\theta(T)$ for this value of $\theta$?

SOLUTION. To find the $\theta$ such that $L(\theta) = .5$ means finding $\lambda_\theta$ such that $\sum_{k=5}^{\infty} p(k; \lambda_\theta T_0) = .5$. Using the tables [5] we see that this means that $\lambda_\theta T_0 = 4.67$. Therefore $\theta = 4{,}180$. From (15) and (16) we find that $E_\theta(r) = 3.97$ and $E_\theta(T) = 424$.

PROBLEM 4. Consider the truncated nonreplacement plan meeting the conditions of Problem 2. For what values of $\theta$ is $L(\theta) = .5$?

SOLUTION. This means finding $p_\theta$ such that $\sum_{k=5}^{\infty} b(k; 42, p_\theta) = .5$. Using the tables [8] this means $p_\theta = .1104$. Since $p_\theta = 1 - e^{-T_0/\theta}$, the appropriate $\theta = 4{,}274$. Here $E_\theta(r)$ and $E_\theta(T)$ will be approximately the same as in the replacement case; they have not been computed exactly.

PROBLEM 5. Find a test of the form $\hat{\theta}_{r,n} > C$, discussed in Section 3, which will have an O.C. curve such that $L(\theta_0) = .95$ when $\theta_0 = 1{,}500$, and $L(\theta_1) \leqq .05$, when $\theta_1 = 500$.

SOLUTION. From Table 1, it is readily verified that $r = 10$. Therefore the acceptance region has according to (32) the form

$$\hat{\theta}_{10,n} > C = \theta_0 \chi^2_{1-\alpha}(20)/20 = 815.$$

PROBLEM 6. Set $n = 20$ in Example 5. Compute $E_\theta(r)$ and $E_\theta(T)$ at $\theta = 1{,}500$ and $\theta = 500$, respectively.

SOLUTION. If we interpret the test as in the second paragraph of Section 3, it may be possible to stop experimentation with fewer than 10 failures and before time $x_{10,20}$. Using formulae (29) and (30) for $\theta = 1500$ we get $E_\theta(\rho) = 5.39$ and $E_\theta(T) = 475$. When $\theta = 500$, $E_\theta(\rho) = 9.93$, and $E_\theta(T) = 331$. It is interesting to note that if $\hat{\theta}_{10,20}$ is computed only after observing $x_{10,20}$, the number of failures would always be 10. Furthermore, the expected waiting time to reach a decision would then be $E_\theta(X_{10,20}) = \theta \sum_{k=1}^{10} (21 - k)^{-1}$. For $\theta = 1{,}500$, $E_\theta(X_{10,20}) = 1{,}004$ and for $\theta = 500$, $E_\theta(X_{10,20}) = 335$. Thus there is considerable saving if we take advantage of continuous availability of information. The ultimate in this direction is a purely sequential procedure which is treated in detail in another paper.

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