ON THE ASYMPTOTIC EFFICIENCY OF CERTAIN NONPARAMETRIC TWO-SAMPLE TESTS

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1. Summary. In this paper the following asymptotic efficiencies are computed for the given two-sample tests against normal alternatives to the null hypothesis:

rank test for location	$3/\pi \cong 95\%$
median test for location	$2/\pi \cong 64\%$
run test for location	0
run test for dispersion	0
square rank test for dispersion	$15/2\pi^2 \cong 76\%$

Also, general expressions for means and variances of some of these test criteria are found for distributions alternative to the null hypothesis.

2. Asymptotic power of a test. Let T_n be a statistic which is a function of n sample observations x_i (with $i = 1, 2, \cdots, n$) from a population with distribution $F(x; \theta)$. Let the mean of T_n be $\mu_n(\theta)$ and the variance of T_n be $\sigma_n^2(\theta)$, and suppose that T_n is asymptotically normally distributed for all θ in a neighborhood of θ_0. Let $h_n(\theta)$ be the power function of T_n for testing the null hypothesis that $\theta = \theta_0$. For large samples

$$h_n(\theta) = P[|T_n - \mu_n(\theta_0)| > k\sigma_n(\theta_0)] \tag{1}$$

$$h_n(\theta) \cong 1 - \int_{\mu_n(\theta_0)-k\sigma_n(\theta_0)}^{\mu_n(\theta_0)+k\sigma_n(\theta_0)} \frac{\exp\{-[y - \mu_n(\theta)]^2/2\sigma_n^2(\theta)\}}{\sqrt{2\pi}\,\sigma_n(\theta)}\,dy, \tag{2}$$

where k is determined by the chosen significance level α of the test.

We suppose further that $\mu_n(\theta)$ and its first two derivatives are of order one in the neighborhood of θ_0 and that $\sigma_n(\theta)$ and its first two derivatives are of order $1/\sqrt{n}$ in the neighborhood of θ_0. Then it is evident from (2) that $h_n(\theta)$ is essentially unity except in a neighborhood of θ_0; of course $h_n(\theta_0) = \alpha$. On evaluating the first two derivatives of $h_n(\theta)$ at θ_0 one finds that the coefficient of the term of order $\sqrt{n}$ vanishes in $h_n'(\theta_0)$, so that it is of order one, and also that the significant term of $h_n''(\theta_0)$ is of order n. Thus in the neighborhood of θ_0 we have essentially (letting ϕ represent the normalized normal density function)

$$h_n(\theta) \cong \alpha + \frac{k\phi(k)}{\sigma_n^2(\theta_0)}\left(\frac{d\mu_n}{d\theta_0}\right)^2 (\theta - \theta_0)^2 \tag{3}$$

when $|\theta - \theta_0| \ll 1/\sqrt{n}$. We are interested in (3) because it enables us to evaluate the asymptotic efficiency of certain tests without having to evaluate the variance of the test criterion under the alternative hypothesis. In fact, if

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T_n^* with mean μ_n^*, variance σ_n^{*2}, and power h_n^* is the best criterion for testing $\theta = \theta_0$, the asymptotic efficiency of T_n is defined to be

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2(\theta_0)} \left(\frac{d\mu_n}{d\theta_0}\right)^2 \bigg/ \frac{1}{\sigma_n^{*2}(\theta_0)} \left(\frac{d\mu_n^*}{d\theta_0}\right)^2, \tag{4}$$

the limiting ratio of $h_n(\theta) - \alpha$ to $h_n^*(\theta) - \alpha$ using (3). This definition is consistent with Fisher's definition of efficient estimators.

For one-sided tests $h_n(\theta)$ is essentially linear instead of parabolic,

$$h_n(\theta) \cong \alpha + \frac{\phi(k)}{\sigma_n(\theta_0)} \frac{d\mu_n}{d\theta_0} (\theta - \theta_0), \tag{5}$$

in the neighborhood of θ_0. Thus the ratio of $h_n(\theta) - \alpha$ to $h_n^*(\theta) - \alpha$ is the square root of (4). For this reason (asymptotically) tests which are not 100% efficient are less unsatisfactory for one-sided tests than for two-sided tests. A more detailed discussion of asymptotic efficiency will be published by Dwass [4]; see also Levene [11] and Noether [13].

[In the application of (4) to the various statistics investigated in this paper, we require something akin to asymptotic normality of the statistic, uniformly in a neighborhood of the null hypothesis. Generally, as pointed out by a referee, proofs existing in the literature of asymptotic normality do not provide this strong a result and the validity of our computations is not completely justified. However, a careful analysis of power functions by Andrews [1] shows how in such cases the usual limiting distributions may be obtained for certain sequences of alternatives to the null hypothesis. Within this framework, at least, the computations are valid.]

In the evaluation of various nonparametric tests to follow, we shall refer to normal distributions so that T^* is well-known. In this sense, the results are quite specialized. However, the intent of the computations is to furnish evidence for a more general evaluation. The choice of the normal distribution puts the nonparametric tests in competition with t and F tests, which are known to be most powerful for that distribution. If a nonparametric test looks bad relative to the t or F test (assuming normality), then one can observe only that the test should not be used when there is assurance of sensible normality. However, a nonparametric test which is found to be nearly 100% efficient relative to the t or F test has much to recommend it—not much is lost by using it even in the case of normality and it is a reasonable presumption that the test will behave approximately as well (as the t or F test) for other distributions. Thus such a test enables one to avoid the assumption of normality at negligible cost when one has little knowledge of the shape of the population distribution.

On the other hand, one must expect that a nonparametric test which compares favorably with the t test, for example, assuming normal distributions, probably will be unsatisfactory for distributions for which the t test is known to be poor.

3. Median test for two samples. The median test [2], [18], [19] of whether two populations have the same location (assuming they are otherwise the same) consists of testing whether the number u of x observations to the left of the joint median z of the two samples differs significantly from half the total number of x observations. This test applied to normal distributions with the same variance will be shown in this section to have an asymptotic efficiency of $2/\pi$; hence it is of much less interest than the rank test (Section 5) in dealing with essentially normal distributions.

We consider samples of m x's and n y's from populations with distributions $F(x)$ and $G(y)$, but to make an unessential simplification we suppose $m + n = 2r + 1$. The joint distribution of u and z is then given by

$$\begin{aligned} \mathrm{H}(u, z) &= m\tbinom{m-1}{u}\tbinom{n}{r-u}F^u(z)[1 - F(z)]^{m-u-1}G^{r-u}(z)[1 - G(z)]^{n-r+u}dF(z) \\ &\quad + n\tbinom{m}{u}\tbinom{n-1}{r-u}F^u(z)[1 - F(z)]^{m-u}G^{r-u}(z)[1 - G(z)]^{n-r+u-1}dG(z), \end{aligned} \tag{6}$$

where the first term arises when z is an x observation and the second when z is a y observation. In this expression we put

$$u = mF(c) + \sqrt{m}v, \qquad z = c + w/\sqrt{m}, \tag{7}$$

where c satisfies

$$mF(c) + nG(c) = (m + n)/2. \tag{8}$$

Then we use Stirling's formula on the factorials, take logarithms, and neglect terms of order $1/\sqrt{m}$ in the usual fashion to find that v and w are asymptotically normally distributed. The quadratic form is found to be

$$\begin{aligned} v^2\left[\frac{1}{F(1 - F)} + \frac{m}{nG(1 - G)}\right] - 2vw\left[\frac{f}{F(1 - F)} - \frac{g}{G(1 - G)}\right] \\ + w^2\left[\frac{f^2}{F(1 - F)} + \frac{ng^2}{mG(1 - G)}\right] \end{aligned} \tag{9}$$

in which f and g are the first derivatives of F and G, and all four functions are evaluated at c; we must assume the derivatives do not vanish.

When $F = G$, the middle term of (9) vanishes so that v and w are independently distributed and we easily find the variance of v to be $n/4(m + n)$ under the null hypothesis. The variance of u, when $F = G$, is thus $mn/4(m + n)$.

Now we put

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \qquad g(x) = \frac{1}{\sqrt{2\pi}}e^{-(x-\xi)^2/2}, \tag{10}$$

to compute the asymptotic efficiency for the normal case. We must first evaluate, to use (4),

$$\frac{d\mu}{d\theta} = \frac{d}{d\xi}mF(c) = mf(c)\frac{dc}{d\xi} \tag{11}$$

at $c = 0$. There is no closed solution of (8) for c in terms of ξ, but we are interested only in the neighborhood of $\xi = 0$ where

$$c = \frac{n}{m+n}\,\xi + O(\xi^2).$$

The value of (11) is then $mn/\sqrt{2\pi}(m+n)$. The asymptotic efficiency of the median test for location for normal populations is therefore

$$\text{(12)}\qquad \frac{1}{mn/4(m+n)}\left(\frac{mn}{\sqrt{2\pi}(m+n)}\right)^2 \bigg/ \frac{1}{1/m+1/n}\left(\frac{d\xi}{d\xi}\right)^2 = \frac{2}{\pi},$$

the same as given by Cochran [3] for the sign test.

4. Rank test for dispersion. Since the rank test (Section 5) is so successful in testing for location differences, it is natural to inquire into the efficiency of a rank test for dispersion. Let m observations from a population distributed by $F(x)$ and n observations from a population distributed by $G(y)$ be ranked from 1 to $m + n$, and let W be the sum of squares of the deviations of the y ranks from the average rank,

$$\text{(13)}\qquad W = \sum_{i=1}^{n}\left(r_i - \frac{m+n+1}{2}\right)^2,$$

where r_i is the rank of the ith (in order of magnitude) y observation. If the x's are dispersed relative to the y's, W will be relatively small.

We first obtain the mean and variance of W when $F = G$ by means of a generating function which automatically performs certain required sums,

$$\text{(14)}\qquad \phi(t, x, y) = \prod_{i=1}^{m+n}\{yt^{[i-(m+n+1)/2]^2} + x\}.$$

The probability $P(W)$ for a particular value of W is the coefficient of t^W in the coefficient of $x^m y^n$ divided by $\binom{m+n}{m}$. Let us denote $m + n$ by s and the exponent of t in (14) by c_i^2. We now differentiate (14) with respect to t to get

$$\text{(15)}\qquad \frac{d\phi}{dt} = \left[\sum_{i=1}^{s}\frac{c_i^2\, yt^{c_i^2-1}}{yt^{c_i^2}+x}\right]\prod_{i=1}^{s}(yt^{c_i^2}+x).$$

Putting $t = 1$ we find the coefficient of $x^m y^n$ in $y(y+x)^{s-1}\sum c_i^2/\binom{s}{m}$ is

$$\text{(16)}\qquad E(W) = n(s+1)(s-1)/12.$$

For the variance, differentiating (15) again with respect to t, putting $t = 1$, etc., we get:

$$\text{(17)}\qquad \begin{aligned} E[W(W-1)] &= \{\tbinom{s-1}{n-1}[\textstyle\sum c_i^4 - \sum c_i^2] + \tbinom{s-2}{n-2}[(\sum c_i^2)^2 - \sum c_i^4]\}/\tbinom{s}{m} \\ &= n(s+1)\{3m(3s^2-7) + 5(s-1)[s(s+1)(n-1) - 12]\}/720. \end{aligned}$$

From this, under the null hypothesis

$$\text{(18)}\qquad \sigma_W^2 = mn(s+1)(s+2)(s-2)/180.$$

Now it is necessary to obtain $E(W)$ when $F \neq G$. We divide the x axis (on which both populations are now assumed to range) into small intervals Δx_α ($\alpha = -\infty, \cdots 0, 1, 2, \cdots \infty$) and suppose the distributions have densities $f(x)$ and $g(x)$ so that the probability an x observation falls in Δx_α is $p_\alpha \cong f(x_\alpha)\Delta x_\alpha$, where x_α is in Δx_α. Similarly for y observations, $q_\alpha \cong g(x_\alpha)\Delta x_\alpha$.

Let i_α be the number of x observations in Δx_α and j_α be the number of y observations in Δx_α (the Δx_α are chosen sufficiently small that the probability of more than one observation in a single interval is negligible). We let Δx_{α_t} denote the interval which contains y_t; then the rank r_t of y_t is

$$r_t = \sum_{\alpha=-\infty}^{\alpha_t} (i_\alpha + j_\alpha). \tag{19}$$

Now let

$$J_\beta = j_\beta \sum_{\alpha=-\infty}^{\beta} (i_\alpha + j_\alpha). \tag{20}$$

It is clear that $J_\beta = 0$ if Δx_β does not contain a y, and is equal to the rank of y if Δx_β does contain a y. We have then

$$\begin{aligned} E(W) &= E\left[\sum_{t=1}^{n}\left(r_t - \frac{m+n+1}{2}\right)^2\right] \\ &= E\left[\sum_{i=1}^{n} r_i^2 - (m+n+1)\sum_{t=1}^{n} r_t + n(m+n+1)^2/4\right] \\ &= E\left[\sum_{-\infty}^{\infty} J_\beta^2 - (m+n+1)\sum_{-\infty}^{\infty} J_\beta + n(m+n+1)^2/4\right]. \end{aligned} \tag{21}$$

In terms of the p_α and q_α

$$\begin{aligned} E(\sum J_\beta) &= \lim_{\Delta x_\alpha \to 0} \sum_\beta nq_\beta \sum_{\alpha=-\infty}^{\beta} mp_\alpha + E(\sum j_\beta^2) + \sum_\beta \sum_{\alpha=-\infty}^{\beta-1} n(n-1) q_\alpha q_\beta \\ &= mn \int_{-\infty}^{\infty} F(x) g(x)\, dx + n + \binom{n}{2}. \end{aligned} \tag{22}$$

$$\begin{aligned} E(\sum J_\beta^2) &= \lim_{\Delta x_\alpha \to 0} E \sum_{\beta=-\infty}^{\infty} j_\beta^2 \left[\sum_{\alpha=-\infty}^{\beta} (i_\alpha + j_\alpha)\right]^2 \\ &= \lim E \sum_\beta j_\beta^2 \sum_{\alpha=-\infty}^{\beta} \sum_{\gamma=-\infty}^{\beta} (i_\alpha i_\gamma + i_\alpha j_\gamma + j_\alpha i_\gamma + j_\alpha j_\gamma) \\ &= \lim E \sum_{\beta,\alpha,\gamma} j_\beta^2 [m^2 p_\alpha p_\gamma + mp_\alpha \delta_{\alpha\gamma} - mp_\alpha p_\gamma \\ &\quad + mp_\alpha j_\gamma + mp_\gamma j_\alpha + j_\alpha j_\gamma] \\ &= \lim \sum_{\beta,\alpha,\gamma} \{(m^2 p_\alpha p_\gamma + mp_\alpha \delta_{\alpha\gamma} - mp_\alpha p_\gamma)[n^2 q_\beta^2 + nq_\beta(1 - q_\beta)] \end{aligned}$$

$$(23)\quad \begin{aligned} &+ 2mp_\alpha [\delta_{\gamma\beta}(2n^{(2)} q_\beta^2 + nq_\beta) + n^{(3)} q_\gamma q_\beta^2 + n^{(2)} q_\gamma q_\beta] \\ &+ n^{(4)} q_\alpha q_\gamma q_\beta^2 + n^{(3)} q_\alpha q_\gamma q_\beta + \delta_{\alpha\gamma}(n^{(4)} q_\alpha^2 q_\beta^2 + n^{(3)} q_\alpha q_\beta^2 \\ &+ n^{(3)} q_\alpha^2 q_\beta + n^{(2)} q_\alpha q_\beta) + 2\delta_{\gamma\beta}(2n^{(3)} q_\alpha q_\beta^2 + n^{(2)} q_\alpha q_\beta) \\ &+ \delta_{\alpha\beta}\delta_{\gamma\beta}(5n^{(3)} q_\beta^3 + 7n^{(2)} q_\beta^2 + nq_\beta)\} \\ = {} & nm^{(2)} \int_{-\infty}^{\infty} F^2\, dG + 3mn \int_{-\infty}^{\infty} F\, dG + 2n^{(2)} m \\ &\cdot \int_{-\infty}^{\infty} FG\, dG + (2n^3 + 3n^2 + n)/6. \end{aligned}$$

Putting the final forms of (22) and (23) in (21) gives

$$(24)\quad \begin{aligned} E(W) = {} & (n/12)[3(s + 1)^2 - 6(n + 1)(s + 1) + 2(n + 1)(2n + 1)] \\ &+ mn\left[(m - 1)\int F^2\, dG + 2(n - 1)\int FG\, dG - (s - 2)\int F dG\right]. \end{aligned}$$

We are now ready to compute the efficiency of W relative to the standard F test using

$$(25)\qquad f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad g(x) = \frac{1}{\sigma} f\left(\frac{x}{\sigma}\right).$$

We assume the asymptotic normality of W uniformly in a neighborhood of $\sigma = 1$ in order to carry out the computation. In the sense of Hoeffding [5], W is a U statistic; hence Hoeffding's general result on the asymptotic normality of U statistics can be applied to W in just the same way as by Lehmann [9] in establishing the normality of the rank test criterion for a fixed parameter value. However, we need more than this and must refer to Andrews [1] for a rigorous discussion of large sample power functions to justify such computations.

On substituting (25) in (24) and differentiating with respect to σ, we obtain

$$(26)\quad \begin{aligned} \frac{dE(W)}{d\sigma} = {} & mn\Big\{(m - 1)\int (x^2 - 1)F\, dF + 2(n - 1)\int (x^2 - 1)F^2\, dF \\ &+ 2(n - 1)\int F\left(\int_{-\infty}^{x} (t^2 - 1)\, dF(t)\right) dF(x) - (s - 2)\int (x^2 - 1)\, F dF\Big\}. \end{aligned}$$

The third integral may be written $(n - 1)\int (x^2 - 1)(1 - F^2)dF$, so that

$$(27)\quad \begin{aligned} \frac{dE(W)}{d\sigma} &= mn(s - 2)\left\{\int (x^2 - 1)F^2\, dF - \int (x^2 - 1)\, F dF\right\} \\ &= mn(s - 2) \,/\, 2\pi\sqrt{3}, \end{aligned}$$

using the evaluation of the integrals given by Jones [7].

For the F test we have $\mu^*(\sigma) = 1/\sigma^2$ and the variance under the null hypothesis is $\sigma_F^2 = 2(m + n)/mn$. Substituting these results together with (18) and (27) into (4), we find the asymptotic efficiency of W to be

$$\frac{180}{mn(m+n)^3}\left[\frac{mn(m+n)}{2\pi\sqrt{3}}\right]^2 \Big/ \frac{mn}{2(m+n)} 2^2 = \frac{15}{2\pi^2}, \tag{28}$$

which is about 76%. For one-sided tests, the square root of this figure is about 87%.

5. Rank test for location. Van der Vaart [16] announced that the Wilcoxon [20] test for location had an asymptotic efficiency of $3/\pi$. However, Pitman [15] seems to have priority[1]. Since there is no readily accessible derivation of this result in print, it is perhaps worth a brief computation here. Using the same notations as before, let U, the test criterion, be the number of times a y precedes an x in an ordered sample of m x's and n y's. Then

$$U = \sum_{\alpha=-\infty}^{\infty} j_\alpha \sum_{\beta=\alpha}^{\infty} i_\beta . \tag{29}$$

Its expectation may be found easily by the method of the previous section. In this case, however, the result is obtained immediately by Mann and Whitney [12] to be

$$E(U) = mn \int_{-\infty}^{\infty} [1 - F(x)]g(x)\, dx. \tag{30}$$

Under the null hypothesis, $\sigma_U^2 = mn(m + n + 1) / 12$. On putting (10) in (30) we find $d\mu/d\xi = -mn/2\sqrt{\pi}$. The asymptotic efficiency is

$$\frac{12}{mn(m+n+1)}\left(\frac{mn}{2\sqrt{\pi}}\right)^2 \Big/ \frac{1}{1/m + 1/n}\left(\frac{d\xi}{d\xi}\right)^2 = \frac{3}{\pi}. \tag{31}$$

Lehmann and Stein [8] have shown that Pitman's [14] randomization test is the most powerful nonparametric test for normal alternatives. Its asymptotic efficiency is shown by Hoeffding [6] to be unity. Dwass [4] has shown that the most powerful rank order test also has unit asymptotic efficiency.

6. Run test for location and dispersion. Pitman [15] found that the Wald and Wolfowitz [17] run test has zero asymptotic efficiency for testing either location or dispersion. These results may be computed at once by putting (10) or (25) in Wolfowitz's [21] expression for the expected total number of runs

$$\int_{-\infty}^{\infty} \frac{2mn\, f(x)\, g(x)}{m\, f(x) + n\, g(x)}\, dx$$

[1] These results are given in lecture notes prepared for a course at Columbia University given in 1948; several copies were distributed to various statisticians in the U.S. but no other copies are available. The author borrowed a copy after being informed of its existence by a referee.

to find $d\mu/d\xi$ or $d\mu/d\sigma$. Then (4) may be used with the t or F test for T_n^*. In this connection, I. R. Savage has raised a question about the proper use of the asymptotic normality of the run criterion; see Lehmann [10].

7. Lehmann's rank test for dispersion. Another test for dispersion described by Lehmann [9] consists of forming all the $\binom{m}{2}$ positive differences between the m observations on x and the $\binom{n}{2}$ positive differences between the n observations on y; the rank test is then applied to these differences. Let V be the number of times x differences exceed y differences, and let the x and y populations have densities $f(x)$ and $g(y)$. A tentative value of $27/4\pi^2$ (about 68%) has been computed for the asymptotic efficiency of V relative to the standard F test, using methods analogous to those of Section 4.

However, V is not completely distribution-free; its distribution depends on the form of $f(x)$ even when $f = g$. The mean is independent of the form

$$E(V \mid f = g) = m(m - 1)n(n - 1)/8,$$

but I have been unable to show that the variance of V or even the large sample variance is independent of $f(x)$. The calculation of the value $27/4\pi^2$ used the unproved assumption that the asymptotic variance is independent of f when $f = g$, in that numerous eight-fold integrals were evaluated using the exponential instead of the normal distribution.

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