To show that this theorem covers a class of stochastic processes of practical interest, it is shown next that the condition (1) of the theorem is true in strictly stationary processes which are normal. For this, it suffices to show that

$$\frac{P[(x > c), (y > c)]}{P(x > c)} \to 0, \qquad (c \to \infty), \tag{2}$$

where $x$ and $y$ have a bivariate normal distribution with means zero, variances unity and covariance $\rho$, with $|\rho| < 1$. Now

$$P[(x > c), (y > c)] = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_c^\infty \int_c^\infty \exp\left[\frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right] dx\, dy.$$

The substitution $x = r/c + c$ and $y = t/c + c$ leads to

$$P[(x > c), (y > c)] = \frac{\exp[(-c^2/(1+\rho)]}{2\pi c^2\sqrt{1-\rho^2}} \int_0^\infty \int_0^\infty \exp\left[-\frac{r^2 - 2\rho rt + t^2}{2c^2(1-\rho^2)}\right] \exp\left[-\frac{r+t}{1+\rho}\right] dr\, dt$$

$$\sim \frac{1}{2\pi} \exp\left(\frac{-c^2}{1+\rho}\right)\left[\frac{(1+\rho)^{3/2}}{\sqrt{1-\rho}}\frac{1}{c^2} - 0\left(\frac{1}{c^4}\right)\right], \qquad c \text{ large}.$$

Since $P(x > c) \sim (1/\sqrt{2\pi}) \exp(-\frac{1}{2}c^2)$, statement (2) follows.

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# EXPRESSION OF THE $k$-STATISTICS $k_9$ AND $k_{10}$ IN TERMS OF POWER SUMS AND SAMPLE MOMENTS

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The $k$ statistics are of interest to workers in the theory of sampling distributions and moment statistics. They are related also to certain aspects of the theory of numbers and combinatory analysis, as indicated by Dressel [1].

The $k$ statistics were introduced by Fisher in 1928 [2] to estimate the cumulants

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or Thiele semi-invariants of a population. Dressel [1] has given a table of the $k_r$ $(r = 1, 2, \cdots, 8)$ in terms of the sums $s_r$ of the $r$th powers of the observations in a sample of size $n$. This note adds $k_9$ and $k_{10}$ to those available in print.

One may readily obtain $k_9$ and $k_{10}$ in terms of the sample moments $m_r$ about the sample mean by replacing $s_1$ by zero and $s_r$ $(r > 1)$ by $nm_r$ in the following expressions.

The expressions for $k_9$ and $k_{10}$ were obtained by following Kendall [3] and using tables of the symmetric functions [4]. The work has been carefully checked. A fundamental check given by Dressel has been successfully applied to both expressions. This check has revealed a correction for $L_6$ as given by Dressel: the coefficient of (5) (1) in $L_6$ should be

$$-15(n^4 + 2n^3 - 7n^2 + 4n).$$

It is found that $n^{(9)}k_9$

$$\begin{aligned}
&= (n^8 + 219n^7 + 3721n^6 + 6189n^5 - 7250n^4 + 2160n^3)s_9 \\
&\quad - 9(n^7 + 219n^6 + 3721n^5 + 6189n^4 - 7250n^3 + 2160n^2)\, s_8s. \\
&\quad - 36(n^7 + 93n^6 + 277n^5 - 1917n^4 + 2746n^3 - 1200n^2)s_7s_2 \\
&\quad + 72(n^6 + 156n^5 + 1999n^4 + 2136n^3 - 2252n^2 + 480n)s_7s_1^2 \\
&\quad - 84(n^7 + 33n^6 - 83n^5 + 543n^4 - 1214n^3 + 720n^2)s_6s_3 \\
&\quad + 504(n^6 + 63n^5 + 97n^4 - 687n^3 + 766n^2 - 240n)s_6s_2s_1 \\
&\quad - 504(n^5 + 94n^4 + 731n^3 + 254n^2 - 240n)s_6s_1^3 \\
&\quad - 126(n^7 + 9n^6 + 61n^5 - 201n^4 + 370n^3 - 240n^2)s_5s_4 \\
&\quad + 1008(n^6 + 21n^5 - 11n^4 + 171n^3 - 422n^2 + 240n)s_5s_3s_1 \\
&\quad + 756(n^6 + 18n^5 - 113n^4 + 198n^3 - 104n^2)s_5s_2^2 \\
&\quad - 4536(n^5 + 34n^4 - 9n^3 - 106n^2 + 80n)s_5s_2s_1^2 \\
&\quad + 3024(n^4 + 49n^3 + 176n^2 - 16n)s_5s_1^4 \\
&\quad + 630(n^6 + 9n^5 + 61n^4 - 201n^3 + 370n^2 - 240n)s_4^2s_1 \\
&\quad + 2520(n^6 - 5n^4 + 4n^2)s_4s_3s_2 \\
&\quad - 7560(n^5 + 10n^4 + 15n^3 - 10n^2 - 16n)s_4s_3s_1^2 \\
&\quad - 11340(n^5 + 6n^4 - 41n^3 + 66n^2 - 32n)s_4s_2^2s_1 \\
&\quad + 30240(n^4 + 14n^3 - 19n^2 + 4n)s_4s_2s_1^3 \\
&\quad - 15120(n^3 + 21n^2 + 20n)s_4s_1^5 \\
&\quad + 560(n^6 - 6n^5 + 31n^4 - 66n^3 + 40n^2)s_3^3 \\
&\quad - 15120(n^5 - 2n^4 + 7n^3 - 22n^2 + 16n)s_3^2s_2s_1 \\
&\quad + 20160(n^4 + 4n^3 + 11n^2 - 16n)s_3^2s_1^3 \\
&\quad - 7560(n^5 - 6n^4 + 11n^3 - 6n^2)s_3s_2^3 \\
&\quad + 90720(n^4 - n^3 - 4n^2 + 4n)s_3s_2^2s_1^2 \\
&\quad - 151200(n^3 + 3n^2 - 4n)s_3s_2s_1^4 \\
&\quad + 60480(n^2 + 6n)s_3s_1^6 + 22680(n^4 - 6n^3 + 11n^2 - 6n)s_2^4s_1 \\
&\quad - 151200(n^3 - 3n^2 + 2n)s_2^3s_1^3 + 272160(n^2 - n)s_2^2s_1^5 \\
&\quad - 181440n\, s_2s_1^7 + 40320\, s_1^9
\end{aligned}$$

Similarly it is found that $n^{(10)}k_{10}$

$$= (n^9 + 466n^8 + 15706n^7 + 72976n^6 - 41171n^5 - 41186n^4 + 45624n^3 - 12096n^2)s_{10}$$
$$- 10(n^8 + 466n^7 + 15706n^6 + 72976n^5 - 41171n^4 - 41186n^3 + 45624n^2 - 12096n)s_9s_1$$
$$- 45(n^8 + 212n^7 + 2428n^6 - 9166n^5 + 859n^4 + 27098n^3 - 33528n^2 + 12096n)s_8s_2$$
$$+ 90(n^7 + 339n^6 + 9067n^5 + 31905n^4 - 20156n^3 - 7044n^2 + 6048n)s_8s_1^2$$
$$- 120(n^8 + 88n^7 + 40n^6 + 526n^5 + 2719n^4 - 18758n^3 + 27480n^2 - 12096n)s_7s_3$$
$$+ 720(n^7 + 150n^6 + 1234n^5 - 4320n^4 + 1789n^3 + 4170n^2 - 3024n)s_7s_2s_1$$
$$- 720(n^6 + 213n^5 + 3845n^4 + 7755n^3 - 5526n^2 + 432n)s_7s_1^3$$
$$- 210(n^8 + 32n^7 + 88n^6 + 734n^5 - 5441n^4 + 17378n^3 - 24888n^2 + 12096n)s_6s_4$$
$$+ 1680(n^7 + 60n^6 + 64n^5 + 630n^4 - 1361n^3 - 690n^2 + 1296n)s_6s_3s_1$$
$$+ 1260(n^7 + 57n^6 - 203n^5 - 465n^4 + 2794n^3 - 3912n^2 + 1728n)s_6s_2^2$$
$$- 7560(n^6 + 89n^5 + 365n^4 - 1385n^3 + 1074n^2 - 144n)s_6s_2s_1^2$$
$$+ 5040(n^5 + 120n^4 + 1235n^3 + 900n^2 - 576n)s_6s_1^4$$
$$- 126(n^8 + 16n^7 + 256n^6 - 1274n^5 + 5959n^4 - 16886n^3 + 24024n^2 - 12096n)s_5^2$$
$$+ 2520(n^7 + 24n^6 + 172n^5 - 270n^4 + 259n^3 + 246n^2 - 432n)s_5s_4s_1$$
$$+ 5040(n^7 + 15n^6 - 101n^5 + 405n^4 - 1196n^3 + 1740n^2 - 864n)s_5s_3s_2$$
$$- 15120(n^6 + 33n^5 + 45n^4 + 255n^3 - 766n^2 + 432n)s_5s_3s_1^2$$
$$- 22680(n^6 + 29n^5 - 135n^4 + 115n^3 + 134n^2 - 144n)s_5s_2^2s_1$$
$$+ 60480(n^5 + 45n^4 + 35n^3 - 225n^2 + 144n)s_5s_2s_1^3$$
$$- 30240(n^4 + 60n^3 + 275n^2)s_5s_1^5$$
$$+ 3150(n^7 + 3n^6 + 31n^5 - 375n^4 + 1264n^3 - 1788n^2 + 864n)s_4^2s_2$$
$$- 9450(n^6 + 17n^5 + 125n^4 - 305n^3 + 594n^2 - 432n)s_4^2s_1$$
$$+ 4200(n^7 - 3n^6 + 25n^5 - 45n^4 - 26n^3 + 48n^2)s_4s_3^2$$
$$- 75600(n^6 + 5n^5 - 15n^4 - 5n^3 + 14n^2)s_4s_3s_2s_1$$
$$+ 100800(n^5 + 15n^4 + 35n^3 - 15n^2 - 36n)s_4s_3s_1^3$$
$$- 18900(n^6 + n^5 - 55n^4 + 215n^3 - 306n^2 + 144n)s_4s_2^3$$
$$+ 226800(n^5 + 10n^4 - 55n^3 + 80n^2 - 36n)s_4s_2^2s_1^2$$
$$- 378000(n^4 + 18n^3 - 19n^2)s_4s_2s_1^4$$
$$+ 151200(n^3 + 25n^2 + 30n)s_4s_1^6$$
$$- 16800(n^6 - 3n^5 + 25n^4 - 45n^3 - 26n^2 + 48n)s_3^3s_1$$
$$- 37800(n^6 - 7n^5 + 25n^4 + 65n^3 + 94n^2 - 48n)s_3^2s_2^2$$
$$+ 302400(n^5 + 5n^3 - 30n^2 + 24n)s_3^2s_2s_1^2$$
$$- 252000(n^4 + 6n^3 + 17n^2 - 24n)s_3^2s_1^4$$
$$+ 302400(n^5 - 5n^4 + 5n^3 + 5n^2 - 6n)s_3s_2^3s_1$$
$$- 1512000(n^4 - 7n^2 + 6n)s_3s_2^2s_1^3$$
$$+ 1814400(n^3 + 4n^2 - 5n)s_3s_2s_1^5$$
$$- 604800(n^2 + 7n)s_3s_1^7$$
$$+ 22680(n^5 - 10n^4 + 35n^3 - 50n^2 + 24n)s_2^5$$

$- 567000(n^4 - 6n^3 + 11n^2 - 6n)s_2^4 s_1^2$

$+ 2268000(n^3 - 3n^2 + 2n)s_2^3 s_1^4$

$- 3175200(n^2 - n)s_2^2 s_1^6$

$+ 1814400n\ s_2 s_1^8$

$- 362880\ s_1^{10}$

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# THE PROBABILITY INTEGRAL OF RANGE FOR SAMPLES FROM A SYMMETRICAL UNIMODAL POPULATION

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**1. Summary.** An asymptotic expression is given for the probability integral of range for samples from a symmetrical unimodal population. Its accuracy is investigated for the case of a normal parent population and for sample sizes from 20 to 100. Over this range errors are small, and by using a correction based on values given below the probability integral can be found with a maximum error of 0.0001. Percentage points of range in the normal case are tabled for $n = 20, 40, 60, 80$ and 100.

**2. The asymptotic expansion.** The parent probability density function $\phi(x)$ is symmetrical about $x = 0$ and its integral from 0 to $x$ is denoted by $\Phi(x)$. The p.d.f. of $w$, the range for a sample of size $n$, is

$$(1) \qquad p(w) = n(n-1)\int_{-\infty}^{\infty} \{\Phi(x) - \Phi(x-w)\}^{n-2}\phi(x)\phi(x-w)\,dx.$$

Integrating with respect to $w$ from $-\infty$ to $w$ gives

$$(2) \qquad F(w) = n\int_{-\infty}^{\infty} \{\Phi(x) - \Phi(x-w)\}^{n-1}\phi(x)\,dx.$$

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