A SIMPLE SEQUENTIAL PROCEDURE FOR TESTING STATISTICAL HYPOTHESES[1]

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Summary. In this paper a simple sequential test is suggested. Distribution of the sample size, its moment generating function, the power function of the test, and the ASN (average sample number) function are obtained. The determination of the set of relative optimum zones for making decisoins is shown to be unique. The existence of a class of sets of absolute optimum zones is proved. The suggested test is shown to be consistent. Some possible applications are discussed and a few numerical efficiencies are calculated.

1. Introduction. Let $\{f(x)\}$ be the class of all continuous pdf's (probability distribution functions) defined over a space S. Let random observations be drawn successively from a population having an unknown continuous pdf $f(x)$. Let the simple hypothesis $H_0:f(x) = f_0(x)$ be tested against a certain alternative or a certain class of alternatives. We shall propose a simple sequentual test procedure and be concerned with the investigation of the properties of the test.

To test the null hypothesis $H_0:f(x) = f_0(x)$, we divide S into three mutually exclusive sets (zones):

S_1 is the zone of preference for acceptance;
S_2 is the zone of indifference;
S_3 is the zone of preference for rejection.

Random observations are drawn successively. At each stage, the number of observations falling in each of the three zones will be counted. Let m_i be the number of observations falling in the zone S_i for $i = 1, 2, 3$ at the mth stage (i.e., after the mth observation has been drawn). Let a and r be two predetermined positive integers. Continue to draw observations as long as $m_1 < a$ and $m_3 < r$. The experiment is discontinued as soon as either $m_1 = a$ or $m_3 = r$. The null hypothesis is accepted if $m_1 = a$, and rejected if $m_3 = r$.

For simplicity, we shall restrict S to be n-dimensional Euclidean space (or a subspace of it) and assume, of course, that the pdf $f(x)$ is continuous in S. However, most of the theorems given in this paper can be extended to more general cases with slight modifications.

2. Fundamental lemma. The principal aim of this section is to prove a lemma which was used for obtaining the moment generating function of the sample size and the power function of the test.

Suppose m, p, and q are positive integers and B, C, and D are positive real

Received: 4/27/53, revised 3/22/54.

[1] This work was supported in part by the Office of Naval Research.

numbers. Let

$$
(2.1)\qquad \begin{aligned} h_m(p, q, B, C, D) &= \sum_{x=0}^{p-1} \frac{(m-1)!}{(q-1)!x!(m-q-x)!} B^q C^x D^m, \\ h(p, q, B, C, D) &= \sum_{m=q}^{\infty} h_m(p, q, B, C, D). \end{aligned}
$$

Then we have the following

LEMMA 1. *If $D < 1$ and $D + CD < 1$, then*

$$
(2.2)\qquad \begin{aligned} &h(p, q, B, C, D) \\ &= \left(\frac{BD}{1 - D - CD}\right)^q \left(1 - \int_0^{CD/(1-D)} \frac{(p+q-1)!}{(p-1)!(q-1)!} z^{p-1}(1-z)^{q-1}\, dz\right). \end{aligned}
$$

PROOF. From definition (2.1) we have

$$
(2.3)\qquad \begin{aligned} h(p, q, B, C, D) &= \sum_{m=q}^{\infty} \sum_{x=0}^{p-1} \frac{(m-1)!}{(q-1)!x!(m-q-x)!} B^q C^x D^m \\ &= \sum_{x=0}^{p-1} \frac{B^q C^x D^{q+x}}{(q-1)!x!} \sum_{m=q+x}^{\infty} \frac{(m-1)!}{(m-q-x)!} D^{m-q-x} \\ &= \sum_{x=0}^{p-1} \frac{B^q C^x D^{q+x}}{(q-1)!x!} (q+x-1)!(1-D)^{-(q+x)} \\ &= \sum_{x=0}^{p-1} \frac{(q+x-1)!}{(q-1)!x!} \left(\frac{BD}{1-D}\right)^q \left(\frac{CD}{1-D}\right)^x \\ &= \left(\frac{BD}{1-D-CD}\right)^q \sum_{x=0}^{p-1} \binom{q+x-1}{x} \\ &\qquad \cdot \left(\frac{1-D-CD}{1-D}\right)^q \left(\frac{CD}{1-D}\right)^x \\ &= \left(\frac{BD}{1-D-CD}\right)^q \\ &\qquad \cdot \left(1 - \int_0^{CD/(1-D)} \frac{(p+q-1)!}{(p-1)!(q-1)!} z^{p-1}(1-z)^{q-1}\, dz\right). \end{aligned}
$$

Thus, Lemma 1 is proved.

3. Distribution of the sample size and its moment generating function. The distribution of the sample size, (a, r, S_1, S_2, S_3) being chosen, depends upon the true underlying distribution, $f(x)$, being tested. In this section, we derive the pdf $g_f(m;\ a, r, S_1, S_3)$ of the sample size m and its mgf (moment generating function) $M_f(t;\ a, r, S_1, S_3)$ under the assumption that $f(x)$ is the true underlying distribution and the set of parameters (a, r, S_1, S_2, S_3) is predetermined.

Throughout this paper, we shall denote by A, I, and R the following three quantities:

$$A = \int_{S_1} f(x)\, dx, \qquad I = \int_{S_2} f(x)\, dx, \qquad R = \int_{S_3} f(x)\, dx. \tag{3.1}$$

We shall denote these quantities by

(a) A_i, I_i, and R_i, if $f(x)$ is replaced by $f_i(x)$ for $i = 0$ or 1;

(b) A', I', and R', if (S_1, S_2, S_3) is replaced by (S_1', S_2', S_3');

(c) A_i', I_i', and R_i', if $f(x)$ is replaced by $f_i(x)$ for $i = 0$ or 1, and (S_1, S_2, S_3) by (S_1', S_2', S_3').

With the definitions (3.1), it is easily seen that the pdf is given by

$$\begin{aligned} g_f(m;\ a, r, S_1, S_3) = \sum_{x=0}^{a-1} \frac{(m-1)!}{(r-1)!x!(m-r-x)!} R^r A^x I^{m-r-x} \\ + \sum_{x=0}^{r-1} \frac{(m-1)!}{(a-1)!x!(m-a-x)!} A^a R^x I^{m-a-x}. \end{aligned} \tag{3.2}$$

Therefore the mgf is given by

$$\begin{aligned} M_f(t;\ a, r, S_1, S_3) = \sum_{m=r}^{\infty} \sum_{x=0}^{a-1} \frac{(m-1)!}{(r-1)!x!(m-r-x)!} R^r A^x I^{m-r-x} e^{mt} \\ + \sum_{m=a}^{\infty} \sum_{x=0}^{r-1} \frac{(m-1)!}{(a-1)!x!(m-a-x)!} A^a R^x I^{m-a-x} e^{mt}. \end{aligned} \tag{3.3}$$

This can be written as

$$\begin{aligned} M_f(t;\ a, r, S_1, S_3) &= \sum_{m=r}^{\infty} \sum_{x=0}^{a-1} \frac{(m-1)!}{(r-1)!x!(m-r-x)!} \left(\frac{R}{I}\right)^r \left(\frac{A}{I}\right)^x (Ie^t)^m \\ &\quad + \sum_{m=a}^{\infty} \sum_{x=0}^{r-1} \frac{(m-1)!}{(a-1)!x!(m-a-x)!} \left(\frac{A}{I}\right)^a \left(\frac{R}{I}\right)^x (Ie^t)^m \\ &= h(a, r, R/I, A/I, Ie^t) + h(r, a, A/I, R/I, Ie^t). \end{aligned} \tag{3.4}$$

Thus, by Lemma 1, the mgf can be written as

$$\begin{aligned} M_f(t;\ a, r, S_1, S_3) = \left(\frac{Re^t}{1-(1-R)e^t}\right)^r \\ \cdot \left(1 - \int_0^{Ae^t/(1-Ie^t)} \frac{(a+r-1)!}{(a-1)!(r-1)!} z^{a-1}(1-z)^{r-1}\, dz\right) \\ + \left(\frac{Ae^t}{1-(1-A)e^t}\right)^a \left(1 - \int_0^{Re^t/(1-Ie^t)} \frac{(r+a-1)!}{(r-1)!(a-1)!} z^{r-1}(1-z)^{a-1}\, dz\right). \end{aligned} \tag{3.5}$$

4. The power and ASN functions. Suppose the set of parameters (a, r, S_1, S_2, S_3) is predetermined. Then, it is easily seen that for any alternative $f(x)$, the power function is given by

$$\varphi(f;\ a, r, S_1, S_3) = \sum_{m=r}^{\infty} \sum_{x=0}^{a-1} \frac{(m-1)!}{(r-1)!x!(m-r-x)!} R^r A^x I^{m-r-x}. \tag{4.1}$$

By Lemma 1, this can be written as

$$
\begin{aligned}
\varphi(f;\quad a, r, S_1, S_3) &= h(a, r, R/I, A/I, I), \\
&= 1 - \int_0^{1-\beta} \frac{(a + r - 1)!}{(a - 1)!(r - 1)!} z^{a-1}(1 - z)^{r-1}\, dz, \\
&= \int_0^{\beta} \frac{(a + r - 1)!}{(a - 1)!(r - 1)!} z^{r-1}(1 - z)^{a-1}\, dz,
\end{aligned}
\tag{4.2}
$$

where $\beta = \beta(f;\quad S_1, S_3) = 1/(1 + A/R)$.

By the use of the mgf (3.5), it is easily verified that the average sample number (ASN) is given by

$$
\begin{aligned}
\mu(f;\quad a, r, S_1, S_3) = &\frac{r}{R}\left[\varphi(f;\quad a, r, S_1, S_3) - \binom{r + a - 1}{r}\beta^r(1 - \beta)^a\right] \\
&+ \frac{a}{A}\left[1 - \varphi(f;\quad a, r, S_1, S_3) - \binom{r + a - 1}{a}\beta^r(1 - \beta)^a\right].
\end{aligned}
\tag{4.3}
$$

This can also be shown to be

$$
\begin{aligned}
\mu(f;\quad a, r, S_1, S_3) = &\frac{r}{R}\left[1 - \int_0^{1-\beta} \frac{(a + r)!}{(a - 1)!r!} z^{a-1}(1 - z)^r\, dz\right] \\
&+ \frac{a}{A}\left[1 - \int_0^{\beta} \frac{(r + a)!}{(r - 1)!a!} z^{r-1}(1 - z)^a\, dz\right].
\end{aligned}
\tag{4.4}
$$

5. Optimum zones S_1, S_2, S_3. In testing the null hypothesis $f_0(x)$ against an alternative hypothesis $f_1(x)$, all the four quantities

$$
\varphi(f_0;\quad a, r, S_1, S_3), \qquad \varphi(f_1;\quad a, r, S_1, S_3),
$$

$$
\mu(f_0;\quad a, r, S_1, S_3), \qquad \mu(f_1;\quad a, r, S_1, S_3)
$$

are functions of the four parameters a, r, S_1, and S_3. Accordingly, there may be many ways of defining the optimum zones. However, in choosing a definition, we should take into consideration the following three problems: (a) the definition itself should be reasonable from the point of view of the statistician; (b) it must be realizable, that is, the optimum zones must exist; and (c) it can be put in a form suitable for applications.

Furthermore, if the pair of positive integers (a, r) is preassigned, a set of optimum zones should be such that it is optimum (in some sense) among all possible sets (S_1, S_2, S_3). If the pair (a, r) is to be determined by the experimenter, then a set should be so chosen that it has certain optimum properties in the whole parameter space $\{(a, r, S_1, S_2, S_3)\}$, that is, it is optimum for all possible choices of pairs (a, r) and all possible choices of sets (S_1, S_2, S_3). In the following, we give two definitions, one for a fixed pair (a, r) and the other for the general case. However, the determination of the optimum zones for the general case is so difficult that we shall just prove their existence.

For any given set (α, φ, a, r), where $0 < \alpha < \varphi < 1$, we shall denote by $\Omega_{\alpha,\varphi,a,r}$ the class of all possible sets of the three zones (S_1, S_2, S_3) which satisfy the following two conditions:

$$\varphi(f_0;\ a, r, S_1, S_3) = \alpha, \qquad \varphi(f_1;\ a, r, S_1, S_3) = \varphi. \tag{5.1}$$

Here, we have assumed that the class $\Omega_{\alpha,\varphi,a,r}$ is nonempty. A proof of the existence of such a class under certain general conditions will be given in Section 6.

A test is said to have the *strength* (α, φ), if its power function satisfies the two conditions (5.1). Thus, every test based on a set of $\Omega_{\alpha,\varphi,a,r}$ has the strength (α, φ).

Definition I. A set (S_1, S_2, S_3) of $\Omega_{\alpha,\varphi,a,r}$ is said to be *relatively optimum* with respect to (a, r), if the inequalities

$$\begin{aligned} &\mu(f_0;\ a, r, S_1, S_3) \leqq \mu(f_0;\ a, r, S_1', S_3'), \\ &\qquad\qquad \mu(f_1;\ a, r, S_1, S_3) \leqq \mu(f_1;\ a, r, S_1', S_3') \end{aligned} \tag{5.2}$$

hold for all sets $(S_1', S_2', S_3') \varepsilon\, \Omega_{\alpha,\varphi,a,r}$. The three zones of a relative optimum set are called *relative optimum zones.*

To determine the relative optimum zones, we need first to prove the following two lemmas.

Lemma 2. *For fixed a and r, the* ASN *function* $\mu(f;\ a, r, S_1, S_3)$ *decreases as either A or R increases.*

Proof. Taking the partial derivatives of the ASN function (4.4) with respect to A and R, we obtain

$$\frac{\partial\mu}{\partial R} = -\frac{r}{R^2}\left[1 - \int_0^{1-\beta} \frac{(a+r)!}{(a-1)!r!} z^{a-1}(1-z)^r\, dz\right], \tag{5.3}$$

$$\frac{\partial\mu}{\partial A} = -\frac{a}{A^2}\left[1 - \int_0^{\beta} \frac{(r+a)!}{(r-a)!a!} z^{r-1}(1-z)^a\, dz\right]. \tag{5.4}$$

Since (5.3) and (5.4) are always negative, Lemma 2 is proved.

Lemma 3. *Suppose* (a) $f_0(x)$ *and* $f_1(x)$ *are continuous,* (b) *for every real number c, the probability measure of the set* $\{x;\ f_1(x)/f_0(x) = c\}$ *under either hypothesis is zero, and* (c) *the set* (S_1, S_2, S_3) *defined by*

$$S_1 = \{x;\ f_1(x)/f_0(x) \leqq k_0\}, \tag{5.5}$$

$$S_2 = \{x;\ k_0 \leqq f_1(x)/f_0(x) \leqq k_1\}, \tag{5.6}$$

$$S_3 = \{x;\ k_1 \leqq f_1(x)/f_0(x)\}, \tag{5.7}$$

where $k_0 \leqq k_1$ *are two constants, belongs to* $\Omega_{\alpha,\varphi,a,r}$.

Then, for any set (S_1', S_2', S_3') *in* $\Omega_{\alpha,\varphi,a,r}$, *we have*

$$A_0' \leqq A_0, \qquad R_0' \leqq R_0, \qquad A_1' \leqq A_1, \qquad R_1' \leqq R_1. \tag{5.8}$$

Proof. In order to prove Lemma 3, it is sufficient to prove (i) if $A_0' \leqq A_0$, then $R_0' \leqq R_0$, $A_1' \leqq A_1$ and $R_1' \leqq R_1$, and (ii) under the given assumptions, the inequality $A_0' \leqq A_0$ holds.

First, assume $A_0' \leqq A_0$. Since both (S_1, S_2, S_3) and (S_1', S_2', S_3') are in $\Omega_{\alpha,\varphi,a,r}$, then, by (4.2) and (5.1), we have

$$\text{(5.9)} \qquad \text{(a)}\quad A_0' R_0' = A_0/R_0, \qquad \text{(b)}\quad A_1' R_1' = A_1 R_1.$$

By (5.9a), $A_0' \leqq A_0$ implies $R_0' \leqq R_0$. By (5.7), $R_0' \leqq R_0$ implies $R_1' \leqq R_1$ (using an argument of the Neyman-Pearson type). Finally, by (5.9b), $R_1' \leqq R_1$ implies $A_1' \leqq A_1$, proving (i).

Next, assume $A_0' > A_0$. Then, by (5.9a), there exists a positive number δ such that

$$\text{(5.10)} \qquad A_0' = A_0 + \delta A_0, \qquad R_0' = R_0 + \delta R_0.$$

Therefore, by (5.5), (5.6), (5.7), we must have

$$\text{(5.11)} \qquad A_1' > A_1 + \delta A_1, \qquad R_1' < R_1 + \delta R_1,$$

which imply that $A_1'/R_1' > A_1/R_1$. Consequently, we obtain

$$\text{(5.12)} \qquad \varphi(f_1;\ a, r, S_1', S_3') < \varphi(f_1;\ a, r, S_1, S_3).$$

This contradicts the assumption that both (S_1, S_2, S_3) and (S_1', S_2', S_3') are members of $\Omega_{\alpha,\varphi,a,r}$. Hence, the inequality $A_0' \leqq A_0$ must hold, proving (ii), which completes the proof of Lemma 3.

THEOREM 1. *Under the conditions given in Lemma* 3, *the set of the relative optimum zones* (S_1, S_2, S_3) *with respect to* (a, r) *for testing the simple hypothesis* $f_0(x)$ *against the alternative hypothesis* $f_1(x)$ *with strength* (α, φ) *is the set determined by* (5.5), (5.6), *and* (5.7)

Theorem 1 follows from Lemmas 2 and 3.

From Theorem 1, it is seen that, for each (α, φ, a, r), the set of relative optimum zones (S_1, S_2, S_3), when it exists, is uniquely determined. In the following, we shall assume that, for every (α, φ, a, r), the set of the relative optimum zones exists. We shall denote by $\Omega_{\alpha,\varphi}$ the class of all possible sets of the three zones such that the corresponding tests will all have strength (α, φ) for testing $f_0(x)$ against $f_1(x)$, that is, $\Omega_{\alpha,\varphi} = \bigcup_{a,r} \Omega_{\alpha,\varphi,a,r}$. To distinguish the sets in $\Omega_{\alpha,\varphi}$ from the sets in $\Omega_{\alpha,\varphi,a,r}$ for some fixed (a, r), we shall write (a, r, S_1, S_2, S_3) as the general set in $\Omega_{\alpha,\varphi}$. We shall also denote by $\Omega_{\alpha,\varphi,0}$ the class of all sets of the relative optimum zones in $\Omega_{\alpha,\varphi}$, that is, all sets (a, r, S_1, S_2, S_3), where, for each pair (a, r), the set (S_1, S_2, S_3) is the set of relative optimum zones with respect to (a, r).

A set (a, r, S_1, S_2, S_3) of $\Omega_{\alpha,\varphi}$ is said to be *comparable* with another set $(a', r', S_1', S_2', S_3')$ of $\Omega_{\alpha,\varphi}$ if either the two inequalities,

$$\text{(5.13)} \qquad \begin{aligned} &\mu(f_0;\ a, r, S_1, S_3) \leqq \mu(f_0;\ a', r', S_1', S_3'),\\ &\qquad\qquad \mu(f_1;\ a, r, S_1, S_3) \leqq \mu(f_1;\ a', r', S_1', S_3'), \end{aligned}$$

hold simultaneously, or the two inequalities,

$$(5.14)\quad \mu(f_0;\quad a, r, S_1, S_3) \geqq \mu(f_0;\quad a', r', S_1', S_3'),\qquad \mu(f_1;\quad a, r, S_1, S_3) \geqq \mu(f_1;\quad a', r', S_1', S_3'),$$

hold simultaneously. Otherwise, they are said to be *noncomparable*. Two comparable sets are said to be *equivalent*, if all four inequalities in (5.13) and (5.14) hold simultaneously.

LEMMA 4. *Given any set* (a, r, S_1, S_2, S_3) *in* $\Omega_{\alpha,\varphi}$, *there is a set* $(a', r', S_1', S_2', S_3')$ *in* $\Omega_{\alpha,\varphi,0}$ *such that the two inequalities* (5.14) *hold simultaneously.*

The proof is trivial, since we can always choose $a' = a$ and $r' = r$.

LEMMA 5. *For any set* (a, r, S_1, S_2, S_3) *in* $\Omega_{\alpha,\varphi}$, *the number of sets* $(a', r', S_1', S_2', S_3')$ *in* $\Omega_{\alpha,\varphi,0}$ *satisfying the two inequalities* (5.14) *is finite.*

PROOF. For any set $(a', r', S_1', S_2', S_3')$ of $\Omega_{\alpha,\varphi,0}$ (or of $\Omega_{\alpha,\varphi}$ in general), the following two inequalities,

$$(5.15)\quad \mu(f_0;\quad a', r', S_1', S_3') \geqq r'\alpha + a'(1-\alpha),\qquad \mu(f_1;\quad a', r', S_1', S_3') \geqq r'\varphi + a'(1-\varphi),$$

must hold. But, for any set (a, r, S_1, S_2, S_3) in $\Omega_{\alpha,\varphi}$, the two quantities $\mu(f_0;\ a, r, S_1, S_3)$ and $\mu(f_1;\ a, r, S_1, S_3)$ are finite. Thus, Lemma 5 follows from the uniqueness of the set of relative optimum zones for each (a', r').

From Lemmas 4 and 5, it is obvious that, for each set (a, r, S_1, S_2, S_3) in $\Omega_{\alpha,\varphi}$, there exists a comparable set $(a^*, r^*, S_1^*, S_2^*, S_3^*)$ in $\Omega_{\alpha,\varphi,0}$ such that the following two inequalities,

$$(5.16)\quad \mu(f_0;\quad a^*, r^*, S_1^*, S_3^*) \leqq \mu(f_0;\quad a, r, S_1, S_3),\qquad \mu(f_1;\quad a^*, r^*, S_1^*, S_3^*) \leqq \mu(f_1;\quad a, r, S_1, S_3),$$

hold for all sets (a, r, S_1, S_2, S_3) in $\Omega_{\alpha,\varphi}$ which are comparable with $(a^*, r^*, S_1^*, S_2^*, S_3^*)$. Denoting by $\Omega^*_{\alpha,\varphi,0}$ the class of all such sets $(a^*, r^*, S_1^*, S_2^*, S_3^*)$ in $\Omega_{\alpha,\varphi}$, we may conclude:

THEOREM 2. *The class* $\Omega^*_{\alpha,\varphi,0}$ *is a subclass of* $\Omega_{\alpha,\varphi,0}$. *Two distinct sets in* $\Omega^*_{\alpha,\varphi,0}$ *are either equivalent or noncomparable.*

The class $\Omega^*_{\alpha,\varphi,0}$ may be called the class of sets of the *absolute optimum zones*. Since there may be many sets of the absolute optimum zones and the determination of any such set is difficult, we shall assume, throughout the remaining part of this paper, that a and r are preassigned and the three zones are chosen according to (5.5), (5.6), and (5.7). We shall also denote by $\varphi(f)$ and $\mu(f)$ the power and the ASN functions of the test if the three zones are so chosen.

We have seen that, for each (α, φ, a, r), the set of the relative optimum zones (S_1, S_2, S_3), when it exists, is uniquely determined. On the other hand, it is easily seen that, for each (α, a, r), there are an infinite number of sets of the relative optimum zones (S_1, S_2, S_3). We shall denote by $\Omega_{\alpha,a,r,0}$ the class of all such sets of the relative optimum zones (S_1, S_2, S_3), that is

$$\Omega_{\alpha,a,r,0} = \{\Omega_{\alpha,\varphi,a,r} \cap \Omega_{\alpha,\varphi,0};\qquad \alpha < \varphi < 1\}.$$

The test based on a preassigned (α, a, r) and a set (S_1, S_2, S_3) will be called the R. O. (relatively optimum) test with respect to (a, r) for fixed level of significance α, or simply the R. O. test, if $(S_1, S_2, S_3) \varepsilon \Omega_{\alpha,a,r,0}$.

6. Consistency and existence of $\Omega_{\alpha,\varphi,a,r}$. In Section 5, we assumed that, for any given set (α, φ, a, r), the class $\Omega_{\alpha,\varphi,a,r}$ is nonempty. This assumption is valid only when $f_0(x)$ and $f_1(x)$ satisfy certain general conditions. On the other hand, the consistency of the R. O. test depends on the existence of such classes.

Suppose a and r are presassigned positive integers. Suppose the hypothesis $f_0(x)$ is to be tested against the alternative hypothesis $f_1(x)$. Again, we shall assume that $f_0(x)$ and $f_1(x)$ are continuous, and, for every real number c, the probability measure of the set $\{x;\ f_1(x)/f_0(x) = c\}$ under either hypothesis is zero. Let

$$(\alpha, \varphi_1), (\alpha, \varphi_2), (\alpha, \varphi_3), \cdots \qquad 0 < \alpha < \varphi_i < 1, \quad i = 1, 2, 3, \cdots, \tag{6.1}$$

be a sequence of pairs of real numbers. Suppose there exists a sequence of sets of the relative optimum zones

$$(S_{11}, S_{21}, S_{31}), (S_{12}, S_{22}, S_{32}), (S_{13}, S_{23}, S_{33}), \cdots, \qquad (S_{1i}, S_{2i}, S_{3i}) \varepsilon \Omega_{\alpha,\varphi_i,a,r}, \ i = 1, 2, 3, \cdots, \tag{6.2}$$

such that the corresponding sequence of R. O. tests will have (6.1) as the sequence of strengths for testing $f_0(x)$ against $f_1(x)$. Let the corresponding sequences of ASN functions be

$$\mu_1(f_i), \mu_2(f_i), \mu_3(f_i), \cdots, \qquad i = 0, 1. \tag{6.3}$$

We shall say that the sequence of R. O. tests is *conditionally consistent*, if for any alternative $f_1(x)$, the sequences of inequalities

$$\mu_1(f_i) < \mu_2(f_i) < \mu_3(f_i) < \cdots, \qquad i = 0, 1, \tag{6.4}$$

imply the sequence of inequalities

$$\varphi_1 < \varphi_2 < \varphi_3 < \cdots. \tag{6.5}$$

This definition is equivalent to

DEFINITION II. The R. O. test is said to be *conditionally consistent*, if, for any fixed level of significance α and any alternative $f_1(x)$, the power function $\varphi(f_1)$ of the R. O. test increases whenever the ASN functions $\mu(f_i)$, for $i = 0$ or 1, increase.

The following two lemmas apply to the R. O. tests.

LEMMA 6. *For a fixed level of significance α and fixed alternative $f_1(x)$, the power function $\varphi(f_1)$ is a monotone increasing function of I_0.*

PROOF. From (4.2), it is evident that in order to prove Lemma 6, it would be necessary and sufficient to prove that, under the given conditions, if (S_1, S_2, S_3) and (S_1', S_2', S_3') are two different sets of the relative optimum zones such that $I_0' > I_0$, then $A_1'/R_1' < A_1/R_1$. Now, since the level of significance α remains fixed, then, by (4.2), the equality (5.9a) holds. Therefore, there exists $0 < \rho < 1$ such that

$$A_0' = \rho A_0, \qquad R_0' = \rho R_0. \tag{6.6}$$

By (5.5) and (5.7), these equalities imply that

$$A_1' < \rho A_1, \qquad R_1' > \rho R_1. \tag{6.7}$$

Consequently, we obtain

$$A_1'/R_1' < A_1/R_1, \tag{6.8}$$

which completes the proof of Lemma 6.

LEMMA 7. *For a preassigned level of significance α, the* ASN *functions $\mu(f_i)$ for $i = 0$ or 1 are monotone increasing functions of I_0.*

PROOF. Increasing I_0 decreases S_1 and S_3, and therefore A_0, R_0, A_1 and R_1. Thus, Lemma 7 follows using Lemma 2.

THEOREM 3. *The* R. O. *test is conditionally consistent.*

This theorem follows directly from Lemmas 6 and 7.

Conditional consistency is a rather weak property. It does not assure us that as the average sample number approaches infinity, the power of the R. O. test approaches one. Hence, a stronger property is desirable.

DEFINITION II′. The R. O. test will be said to be *absolutely consistent*, if, for every fixed level of significance α and every given alternative $f_1(x)$, the power function $\varphi(f_1)$ tends to 1 as the ASN function $\mu(f_1)$ tends to ∞.

Although the R. O. test is conditionally consistent, it may not be absolutely consistent. We shall verify this assertion by an example. But first, let us state an obvious but useful lemma.

LEMMA 8. *For a fixed level of significance α, if (S_1, S_2, S_3) is a set of relative optimum zones and if (S_1', S_2', S_3') is any other set of the three zones such that $I_0' = I_0$, then we have*

$$\varphi(f_1;\ a, r, S_1, S_3) \geqq \varphi(f_1;\ a, r, S_1', S_3'). \tag{6.9}$$

This lemma is obviously true by (4.2), (5.5), (5.6), (5.7) and (5.9a).

The following example shows that the R. O. test is conditionally consistent, but not absolutely consistent. Let a class of pdf's be given as follows:

$$f(x) = \theta + 2(1 - \theta)x, \qquad 0 \leqq \theta \leqq 1, \quad 0 < x < 1. \tag{6.10}$$

Let the hypothesis

$$H_0: \theta = 1 \tag{6.11}$$

be tested against the alternative hypothesis

$$H_1: \theta = \theta_1, \qquad 0 < \theta_1 < 1. \tag{6.12}$$

Clearly, this is equivalent to testing the uniform density $f_0(x) = 1$ against the alternative $f_1(x) = \theta_1 + 2(1 - \theta_1)x$. Since the ratio $f_1(x)/f_0(x) = f_1(x)$ is a monotone increasing function of x, then any set of the relative optimum zones will have the form $S_1 = (0, x)$, $S_2 = (x, x')$, and $S_3 = (x', 1)$. Furthermore, since, for fixed α, both S_1 and S_3 must satisfy the equality

$$R_0 = \lambda A_0, \tag{6.13}$$

where λ is determined so that $\varphi(f_0) = \alpha$, then x and x' must satisfy

$$x' = 1 - \lambda x. \tag{6.14}$$

As a result, we obtain, by (4.2),

$$\beta_1 = 1/(1 + A_1/R_1) = [\lambda(2 - \theta_1) - \lambda^2(1 - \theta_1)x]/[\theta_1 + \lambda(2 - \theta_1) + (1 - \theta_1)(1 - \lambda^2)x]. \tag{6.15}$$

Taking the limit on β_1 in (6.15), we obtain

$$\lim_{I_0 \to 1} \beta_1 = \lim_{x \to 0} \beta_1 = \lambda(2 - \theta_1)/(\theta_1 + \lambda(2 - \theta_1)) = \beta^* < 1. \tag{6.16}$$

Hence, we have

$$\max_{0 \leqq I_0 < 1} \varphi(f_1) \leqq \varphi^*, \qquad \varphi^* = \int_0^{\beta^*} \frac{(a + r - 1)!}{(a - 1)!(r - 1)!} z^{r-1}(1 - z)^{a-1}\, dz < 1. \tag{6.17}$$

Thus, by Lemma 8, for a given set (α, φ, a, r) with $\varphi > \varphi^*$, the class $\Omega_{\alpha,\varphi,a,r}$ is empty, that is, for the given pair (a, r), there is no set of the three zones giving strength (α, φ) for testing $f_0(x)$ against $f_1(x)$. Therefore, the R. O. test can not be absolutely consistent, though it is always conditionally consistent.

In the following, we give a necessary and sufficient condition for the existence of $\Omega_{\alpha,\varphi,a,r}$ for an arbitrary set (α, φ, a, r) and also a necessary and sufficient condition for the absolute consistency of the R. O. test.

THEOREM 4. *A necessary and sufficient condition for the existence of* $\Omega_{\alpha,\varphi,a,r}$ *is that there exists a number* $k > 0$ *such that* (A) *the probability measure of the set* $w_1 = \{x;\ f_1(x)/f_0(x) \leqq k\beta_0/(1 - \beta_0)\}$ *is positive under either hypothesis and* (B) *the probability measure of the set* $w_3 = \{x;\ f_1(x)/f_0(x) \geqq k\beta_1/(1 - \beta_1)\}$ *is positive under either hypothesis, where* $\beta_i = 1/(1 + A_i/R_i)$ *for* $i = 0$ *or* 1 *are determined from* (4.2) *so that the* R. O. *test would have strength* (α, φ) *for testing* $f_0(x)$ *against* $f_1(x)$.

PROOF. i) Sufficiency. Since the power functions are continuous under the assumptions, then, from Lemmas 6 and 8, it is clear that in order to prove the existence of $\Omega_{\alpha,\varphi,a,r}$, it would be sufficient to prove the existence of $\Omega_{\alpha,\varphi',a,r}$, where $\varphi' \geqq \varphi$, that is, it is sufficient to show that we can find a set of relative optimum zones (S_1', S_2', S_3') such that the following are satisfied:

$$\text{(a)}\ A_0'/R_0' = (1 - \beta_0)/\beta_0, \qquad \text{(b)}\ A_1'/R_1' \leqq (1 - \beta_1)/\beta_1. \tag{6.18}$$

Now, if conditions (A) and (B) hold, we can choose a subset $S_1' \subset w_1$ and a subset

$S_3' \subset w_3$ such that (6.18a) is satisfied. Consequently, we have

$$A_1' \leqq A_0' k\beta_0/(1 - \beta_0), \qquad R_1' \geqq R_0' k\beta_1/(1 - \beta_1). \tag{6.19}$$

Therefore, the inequality (6.18b) holds.

ii) Necessity. Conversely, if the class $\Omega_{\alpha,\varphi,a,r}$ exists, then, by Lemmas 6 and 8, a set of relative optimum zones (S_1'', S_2'', S_3'') exists such that

$$A_0''/R_0'' = (1 - \beta_0)/\beta_0, \qquad A_1''/R_1'' \leqq (1 - \beta_1)/\beta_1 \tag{6.20}$$

hold. Consequently, there exists a number $k > 0$ such that we have

$$A_1''/A_0'' \leqq k\beta_0/(1 - \beta_0), \qquad R_1''/R_0'' \geqq k\beta_1/(1 - \beta_1). \tag{6.21}$$

Therefore, there exist subsets $w_1 \subset S_1''$ and $w_3 \subset S_3''$ such that conditions (A) and (B) are true.

THEOREM 5. *A necessary and sufficient condition for the absolute consistency of the* R. O. *test is that at least one of the following two conditions is true*:

(A′) *for every positive* ϵ, *the probability measure of the set*

$$w_1' = \{x;\ f_1(x)/f_0(x) \leqq \epsilon\}$$

is positive under either hypothesis;

(B′) *for every positive* ϵ', *the probability measure of the set*

$$w_3' = \{x;\ f_1(x)/f_0(x) \geqq \epsilon'\}$$

is positive under either hypothesis.

PROOF. i) Sufficiency. By (4.4), it is seen that the ASN function $\mu(f_1)$ tends to infinity only if at least one of the two quantities A_1 and R_1 tends to zero. Hence, by (4.2), it is obvious that in order to prove the sufficiency it would be sufficient to show that, for every given level of significance α and every alternative $f_1(x)$, the ratio A_1/R_1 tends to zero as R_1 tends to zero. Clearly, for a fixed α, the ratio A_0/R_0 remains fixed. Let

$$A_0/R_0 = d, \tag{6.22}$$

where d is a constant. Then, by (5.7), $R_1 \to 0$ implies $R_0 \to 0$. By (6.22), $R_0 \to 0$ implies $A_0 \to 0$. Finally, by (5.5), $A_0 \to 0$ implies $A_1 \to 0$. Furthermore, if (A′) is true, then

$$\lim_{R_1 \to 0} R_1/R_0 > 1, \qquad \lim_{R_1 \to 0} A_1/A_0 = 0. \tag{6.23}$$

If (B′) is true, then

$$\lim_{R_1 \to 0} R_1/R_0 = \infty, \qquad \lim_{R_1 \to 0} A_1/A_0 < 1. \tag{6.24}$$

Consequently, in either case, we obtain

$$\lim_{R_1 \to 0} A_1/R_1 = \lim_{R_1 \to 0} d(A_1/R_1)(R_0/A_0) = d \lim_{R_1 \to 0} (A_1/A_0)/(R_1/R_0) = 0. \tag{6.25}$$

ii) Necessity. The necessity can be easily proved by contradiction. Assume both (A′) and (B′) are not true. Then, the ratio $f_1(x)/f_0(x)$ must be bounded. Let

$$L = \text{g.l.b. } \{f_1(x)/f_0(x)\}, \qquad U = \text{l.u.b. } \{f_1(x)/f_0(x)\}. \tag{6.26}$$

Choose a set (α, φ, a, r) such that

$$\beta_0/(1 - \beta_0) < L, \qquad \beta_1/(1 - \beta_1) > U, \tag{6.27}$$

where, again, β_0 and β_1 are determined from (4.2) so that the R. O. test should have strength (α, φ). Then, we can not find a $k > 0$ such that (A) and (B) in Theorem 4 hold simultaneously and hence the class $\Omega_{\alpha,\varphi,a,r}$ is empty. This contradicts the assumption that the R. O. test is absolutely consistent. Thus, the necessity of either (A′) or (B′) is established.

From Theorem 5, it is seen that unless (A′) or (B′) is satisfied, the R. O. test can not be absolutely consistent. However, for practical purposes one may modify the procedure and thus obtain a R. O. test with a specified strength (α, φ). The following are two of the possible modifications:

(a) Increasing a and/or r. The power function is in the form of the incomplete beta function. Thus, for an arbitrary pair (α, φ), it may be possible, by increasing a and/or r, to decrease the difference between β_0 and β_1 so that, for some $k > 0$, the conditions (A) and (B) in Theorem 4 are satisfied.

(b) Taking the observations in groups. When observations are taken in groups of size n, one may apply the R. O. test on some appropriate statistic so that the R. O. test will have the specified strength (α, φ). This is because sometimes for some appropriate n, the pdf's of the statistic under the null and the alternative hypotheses may satisfy the condition in Theorem 4. Usually, this is true when n is sufficiently large.

7. Applications. The R. O. test procedure may have a wide variety of applications. In testing a simple hypothesis, the procedure is applicable whenever the pdf under the null hypothesis and the ratio of the pdf's under both the null and the alternative hypotheses are determinable, especially when the condition in Theorem 5 is also satisfied. For example, let $n(x; \theta, \sigma^2)$ be the pdf of a normal distribution, that is,

$$n(x; \theta, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x - \theta)^2\right), \qquad -\infty < x < \infty,$$

where σ^2 is known, and let the hypothesis $H_0{:}\theta = \theta_0$ be tested against the alternative hypothesis $H_1{:}\theta > \theta_0$. For $\theta > \theta_0$, the ratio $n(x; \theta, \sigma^2)/n(x; \theta_0, \sigma^2)$ is a monotone increasing function of x, and both (A′) and (B′) in Theorem 5 are satisfied. Hence we can apply the R. O. test by taking the three intervals $(-\infty, x_1)$, (x_1, x_2), and (x_2, ∞) as S_1, S_2, and S_3, where x_1 and x_2 are determined so that, for fixed "a" and "r", the R. O. test will have a preassigned strength (α, φ) for testing θ_0 against some alternative θ_1 where $\theta_1 > \theta_0$. The determination of x_1 and x_2 can easily be made by trial and error, since x_1 is a monotone decreasing function of φ and x_2 is a monotone increasing function of φ, for a fixed level of significance α. For instance, if $(\alpha, \varphi, a, r) = (.05, .95, 1, 1)$. then, x_1 and x_2 should be so determined that the following two equalities are satisfied:

$$1/(1 + A_0/R_0) = .05, \qquad 1/(1 + A_1/R_1) = .95, \tag{7.1}$$

where A_0, A_1, R_0 and R_1 are given by

$$A_i = \int_{-\infty}^{x_1} n(x; \theta_i, \sigma^2)\, dx, \qquad R_i = \int_{x_2}^{\infty} n(x; \theta_i, \sigma^2)\, dx. \qquad i = 0, 1, \tag{7.2}$$

Thus, if $\theta_1 = \theta_0 + 2\sigma$, then, by the use of the normal probability table, the approximate values of x_1 and x_2 are found to be $x_1 = \theta_0 + .093\sigma$ and $x_2 = \theta_0 + 1.907\sigma$.

If a composite hypothesis is to be tested, sometimes one may also apply the procedure if it is possible to take the observations in groups and a similar region can be found. For instance, the central t-distribution is used in testing the location of the mean of a normal distribution with unknown variance and the χ^2 distribution will be used in testing the variance of a normal distribution with unknown mean.

The following two examples illustrate the application of the test to the nonparametric and multisample problems.

EXAMPLE 1. (Test of the location of the median of a population.) To test whether the median ν of a population is equal to or greater than ν_0, one can take the observations in groups of size n and call an observation 0 if it is less than ν_0 and 1 otherwise. Under the null hypothesis, the sum X of the observations has the binomial density $f(\binom{x}{x}) = \binom{n}{x}(\frac{1}{2})^n$, for $x = 0, 1, 2, \cdots, n$. By grouping the $n + 1$ points $(0, 1, 2, \cdots, n)$ into three different zones, the proposed test is applicable.

EXAMPLE 2. (Comparison of two populations.) Suppose $X_1 < X_2 < \cdots < X_n$ and $Y_1 < Y_2 < \cdots < Y_m$ are the ordered results of two random samples from populations having continuous cumulative distribution functions $F(x)$ and $G(x)$ respectively. Let $s_1, s_2, \cdots, s_n$ be the ranks of the observations of X. Let $W = s_1 + s_2 \cdots + s_n$. Denote by $h(x)$ the pdf of the random variable W. Let the hypothesis $H_0{:}F(x) = G(x)$ be tested against the alternative hypothesis $H_1{:}F(x) > G(x)$. Then, since the density $h_0(x)$ of W under H_0 is known, one may apply the test procedure as follows. Choose two positive integers a and r. Decide on two numbers w' and w'' such that

$$\begin{aligned} \Pr(W < w' \mid H_0) = A_0, \qquad \Pr(w' \leqq W \leqq w'' \mid H_0) &= I_0, \\ \Pr(W > w'' \mid H_0) &= R_0, \end{aligned} \tag{7.3}$$

and such that the pair $(\varphi(f_0), \mu(f_0))$ satisfies certain conditions. Continue to draw samples of sizes (n, m). At each stage, count the number of times that $W < w'$, $w' \leqq W \leqq w''$ and $W > w''$. Denote these numbers by c_1, c_2 and c_3. Then, the proposed test is applicable.

We note that the procedures used in Examples 1 and 2 are not necessarily optimal. They are given here as possible applications of the proposed procedure in general.

8. Efficiency. In this section, we shall investigate the power efficiency of the R. O. test as compared with Wald's sequential probability ratio test.

Let $N(x;\ \theta, \sigma^2)$ be a cumulative normal distribution with an unknown mean θ and known variance σ^2. Let the hypothesis $H_0:\theta = \theta_0$ be tested against an alternative hypothesis $H_1:\theta = \theta_1$. We shall calculate the numerical efficiencies of the R. O. test for the five cases: $\theta_1 = \theta_0 + \lambda\sigma$, $\lambda = 1.0, 1.5, 2.0, 2.5, 3.0$. For each λ, we shall denote by $\psi(\theta)$ and $\eta(\theta)$ the power and the ASN functions of Wald's sequential probability ratio test, and by $\varphi_i(\theta)$ and $\mu_i(\theta)$ the power and ASN functions of the R. O. test for $i = 1, 2$, where by $i = 1$, it is meant $a = r = 1$ and similarly by $i = 2$ is meant $a = r = 2$. Furthermore, let (.05, .95) be the preassigned strength of all the tests, that is, for each $\lambda(\lambda = 1.0, 1.5, 2.0, 2.5, 3.0)$, we have $\psi(\theta_0) = \varphi_i(\theta_0) = .05$ and $\psi(\theta_0 + \lambda\sigma) = \varphi_i(\theta_0 + \lambda\sigma) = .95$ $(i = 1, 2)$. Then, it is obvious that for any real θ, the functions $\psi(\theta)$, $\eta(\theta)$, $\varphi_i(\theta)$ and $\mu_i(\theta)$ $(i = 1, 2)$ depend not only on $\xi = (\theta - \theta_0)/\sigma$, but also on λ (i.e., on H_1). In Tables I and II are given the numerical values of these functions for $\lambda = 1.0, 1.5, 2.0, 2.5, 3.0$ and selected values of ξ. Since the power curves for both the sequential probability ratio and the R. O. tests are close to each other

TABLE I

ξ	Sequential Probability Ratio Test		R. O. Test					
			$a = r = 1$			$a = r = 2$		
	$\psi(\theta)$	$\eta(\theta)$	$\varphi_1(\theta)$	$\mu_1(\theta)$	$\mathcal{E}_1$	$\varphi_2(\theta)$	$\mu_2(\theta)$	$\mathcal{E}_2$
0	.0500	5.2997	.0500	57.4197	.0923	.0500	10.4343	.5079
.50	.5000	8.6695	.5000	117.4618	.0738	.5000	14.9957	.5781
1.00	.9500	5.2997	.9500	57.4197	.0923	.9500	10.4343	.5079
0	.0500	2.3554	.0500	4.2996	.5478	.0500	3.4738	.6780
.50	.2726	3.5711	.2733	7.1090	.5023	.2840	4.4255	.8069
.75	.5000	3.8531	.5000	7.7662	.4961	5000	4.6081	.8362
1.00	.7274	3.5711	.7267	7.1090	.5023	.7160	4.4255	.8069
1.50	.9500	2.3554	.9500	4.2996	.5478	.9500	3.4738	.6780
2.00	.9927	1.5473	.9929	2.5203	.6139	.9951	2.6966	.5738
0	.0500	1.3249	.0500	1.7689	.7490	.0500	2.4051	.5509
.50	.1866	1.8455	.1890	2.3713	.7783	.2018	2.6881	.6865
1.00	.5000	2.1674	.5000	2.7442	.7898	.5000	2.8308	.7656
1.50	.8134	1.8455	.8110	2.3713	.7783	.7982	2.6881	.6865
2.00	.9500	1.3249	.9500	1.7689	.7490	.9500	2.4051	.5509
2.50	.9881	.9581	.9890	1.3671	.7008	.9922	2.1816	.4392
3.00	.9972	.7320	.9979	1.1566	.6329	.9992	2.0649	.3545

TABLE II

ξ	Sequential Probability Ratio Test		R. O. Test $a = r = 1$		$\mathcal{E}_1$
	$\psi(\theta)$	$\eta(\theta)$	$\varphi_1(\theta)$	$\mu_1(\theta)$	
0	.0500	.8479	.0500	1.2241	.6927
.50	.1460	1.1119	.1512	1.4095	.7889
1.00	.3569	1.3484	.3617	1.5770	.8550
1.25	.5000	1.3871	.5000	1.6040	.8648
1.50	.6431	1.3484	.6383	1.5770	.8550
2.00	.8540	1.1119	.8488	1.4095	.7889
2.50	.9500	.8479	.9500	1.2241	.6927
3.00	.9840	.6515	.9862	1.1007	.5919
0	.0500	.5889	.0500	1.0452	.5634
.50	.1232	.7396	.1324	1.0874	.6802
1.00	.2726	.8928	.2860	1.1318	.7888
1.50	.5000	.9633	.5000	1.1519	.8363
2.00	.7274	.8928	.7140	1.1318	.7888
2.50	.8768	.7396	.8676	1.0874	.6802
3.00	.9500	.5889	.9500	1.0452	.5634
3.50	.9807	4718	.9846	1.0185	.4632
4.00	.9927	.3868	.9961	1.0060	.3845

in all the cases considered, then

$$\mathcal{E}_1 = \eta(\theta)/\mu_1(\theta), \qquad \mathcal{E}_2 = \eta(\theta)/\mu_2(\theta), \tag{8.1}$$

as given in the tables can be regarded as the approximate power efficiencies.

From Tables I and II, we observe the following:

(a) In order to obtain high efficiencies, it seems that, when both types of error are fixed and the difference $(\theta_1 - \theta_0)/\sigma$ is small, one should make a and r large.

(b) Some of the figures in the tables are misleading. It is clearly true that no matter which procedure is used, one has to take at least one observation before a decision can be made. Hence, the ASN in either case must be at least one. However, some of the figures for Wald's case are less than one, which can not be regarded as practical. Therefore, in the case $\theta_1 = \theta_0 + 3\sigma$, the efficiencies will be at least .87 uniformly if we assume that ASN is at least one.

(c) If one is interested in improving the efficiency, say, for testing the hypothesis $H_0:\theta = \theta_0$ against the alternative hypothesis $H_1':\theta = \theta_0 + \frac{1}{2}\sigma$, then one may take the observations in groups of size 25 and apply the R. O. test on the means $\bar{x}$ (using $a = r = 1$). In other words, one is now testing the same null hypothesis H_0 against an equivalent alternative hypothesis $H_1'':\theta = \theta_0 + 2.5\sigma_{\bar{x}}$. Consequently, the efficiencies are raised to at least 69 per cent for all alternatives θ between θ_0 and $\theta_0 + \frac{1}{2}\sigma$.

The author wishes to thank the referee for the helpful comments.

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