ON THE MAXIMUM NUMBER OF CONSTRAINTS OF AN ORTHOGONAL ARRAY[1]

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1. Summary. R. C. Bose and K. A. Bush [1] showed how to make use of the maximum number of points, no three of which are collinear, in finite projective spaces for the construction of orthogonal arrays. In particular this enabled them to construct an orthogonal array (81, 10, 3, 3). They proved on the other hand that, in the case considered, the maximum number of constraints does not exceed 12 [1], (Theorem 2C). Hence they state: "We do not know whether we can get 11 or 12 constraints in any other way."

This paper shows first that a 10-rowed orthogonal array, constructed by the geometrical method, cannot be extended to an 11-rowed orthogonal array. It then shows that the number of constraints does not exceed 11. The problem of construction of an orthogonal array with 11 constraints remains unsolved.

This summary should serve, as well, as a correction to the statement made in the abstract, "A remark on the geometrical method of construction of an orthogonal array," published in the *Annals of Mathematical Statistics*, Vol. 25 (1954), p. 177–178, which claimed the nonexistence of an orthogonal array of 11 constraints also.

2. Introduction. The proof is based on an algebraic property of orthogonal arrays, pointed out by Bose and Bush [1]. Let n_{ij}^k denote the number of columns belonging to an array consisting of k rows that have j coincidences (j elements equal) with the ith column. A necessary condition for an array $(\lambda s^t, k, s, t)$ to be orthogonal is that whatever be the number h such that $0 \leqq h \leqq t$, the following equalities hold

$$\sum_{j=0}^{k} n_{ij}^k C_h^j = C_h^k (\lambda s^{t-h} - 1), \qquad i = 1, 2, \cdots, \lambda s^t$$

where the C's are binomial coefficients.

In the case considered, the equalities become, for $i = 1, 2, \cdots, 81$,

$$(1) \quad \begin{aligned} &\sum_{j=0}^{k} n_{ij}^k = 80 & &\sum_{j=0}^{k} j(j-1)\, n_{ij}^k = 8k(k-1) \\ &\sum_{j=0}^{k} j\, n_{ij}^k = 26k & &\sum_{j=0}^{k} j(j-1)(j-2)\, n_{ij}^k = 2k(k-1)(k-2). \end{aligned}$$

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Equations (1) will be used throughout the proof in the special case of $i = 1$. Moreover, it will be assumed, without loss of generality, that the first column consists of zeros only.

3. Derivations.

LEMMA 1. *An orthogonal array* (81, 10, 3, 3), *with* 10 *constraints and* $n_{1j}^{10} = 0$ *for* $j \geqq 5$ *cannot be extended to an* 11-*rowed orthogonal array.*

PROOF. The third and fourth equations of (1) becomes in this case

$$2n_{12}^{10} + 6n_{13}^{10} = 720 - 12n_{14}^{10}$$
$$6n_{13}^{10} = 1440 - 24n_{14}^{10}.$$

Since $n_{13}^{10} \geqq 0$ and $n_{12}^{10} \geqq 0$, both equations imply $n_{14}^{10} \leqq 60$. On the other hand, subtracting the second equation from the first gives $n_{14}^{10} \geqq 60$. Consequently $n_{14}^{10} = 60$ and $n_{12}^{10} = n_{13}^{10} = 0$. Then the second and first equations of (1) imply $n_{11}^{10} = 20$ and $n_{10}^{10} = 0$, respectively. It remains to show now that an array satisfying this solution cannot be extended to an 11-rowed orthogonal array. We observe first that this solution implies $n_{10}^{11} = n_{13}^{11} = 0$ and thus all other n's are uniquely determined, namely $n_{15}^{11} = 9$, $n_{14}^{11} = 60$, $n_{12}^{11} = -10$, $n_{11}^{11} = 21$, which is clearly impossible.

THEOREM 1. *A* 10-*rowed orthogonal array constructed by the method employed by Bose and Bush* [1] *cannot be extended to an* 11-*rowed orthogonal array.*

PROOF. Since the maximum number of points of which no three are collinear in the projective plane considered is equal to four, it is easy to see that the method of construction cannot lead to a column having more than four coincidences with the first column. Hence the theorem is an immediate consequence of Lemma 1.

LEMMA 2. *The number of coincidences between any two columns of a* 6-*rowed orthogonal array is less than six, provided that* $\lambda = s = t = 3$.

PROOF. Consider equations (1) for $k = 6$ and $i = 1$. Express n_{10}^{6}, n_{11}^{6}, n_{12}^{6}, and n_{13}^{6} in terms of n_{14}^{6}, n_{15}^{6}, and n_{16}^{6}. The equations become

$$n_{10}^{6} = 4 + n_{14}^{6} + 4n_{15}^{6} + 10n_{16}^{6}, \qquad n_{11}^{6} = 36 - 4n_{14}^{6} - 15n_{15}^{6} - 36n_{16}^{6},$$
$$n_{12}^{6} = 6n_{14}^{6} + 20n_{15}^{6} + 45n_{16}^{6}, \qquad n_{13}^{6} = 40 - 4n_{14}^{6} - 10n_{15}^{6} - 20n_{16}^{6}.$$

If $n_{16}^{6} > 0$, all the n's are uniquely determined, namely

$$n_{16}^{6} = 1, \quad n_{15}^{6} = n_{14}^{6} = 0, \quad n_{13}^{6} = 20, \quad n_{12}^{6} = 45, \quad n_{11}^{6} = 0, \quad n_{10}^{6} = 14.$$

This means that every 4-rowed subarray must satisfy the equality $n_{14}^{4} = 1$, and again all n's will be uniquely determined as $n_{13}^{4} = 4$, $n_{12}^{4} = 30$, $n_{11}^{4} = 28$, and $n_{10}^{4} = 17$. Hence, if we delete from the original array any two rows, the number of columns which have no coincidences with the first column will increase by three. Since $n_{11}^{6} = 0$, every pair of rows will contain exactly six zeros, contained in three columns each having two zeros. Moreover, different pairs must contain zeros belonging to different such columns. The number of pairs of rows is equal to

20. Hence n_{12}^{6} would have to be equal to 60, contrary to the assumption that $n_{12}^{6} = 45$. This proves Lemma 2.

COROLLARY. *Any orthogonal array* (81, k, 3, 3) *satisfies the equalities* $n_{ij}^{k} = 0$, *provided that* $j \geqq 6$ *and* $1 \leqq i \leqq 81$.

LEMMA 3. *An orthogonal array with* 11 *constraints satisfies one of the nine solutions*

Solution	n_{ij}					
	n_{15}^{11}	n_{14}^{11}	n_{13}^{11}	n_{12}^{11}	n_{11}^{11}	n_{10}^{11}
I	11	55	0	0	11	3
II	12	52	2	2	8	4
III	13	49	4	4	5	5
IV	13	50	0	10	1	6
V	14	45	0	10	6	5
VI	14	46	6	6	2	6
VII	15	42	12	2	3	6
VIII	16	39	14	4	0	7
IX	17	35	20	0	1	7

PROOF. In view of Lemma 2, equations (1) imply

$$2n_{12}^{11} + 3n_{13}^{11} = -110 + 10n_{15}^{11}, \qquad 3n_{11}^{11} + 2n_{12}^{11} = 88 - 5n_{15}^{11}.$$

Hence, $11 \leqq n_{15}^{11} \leqq 17$, since all the n's are nonnegative. All solutions which could be satisfied by an orthogonal array with 11 constraints can now be easily enumerated. For instance, we may express n_{10}^{11}, n_{11}^{11}, n_{12}^{11}, and n_{13}^{11} in terms of n_{14}^{11} and n_{15}^{11}. Then

$$n_{10}^{11} = -96 + n_{14}^{11} + 4n_{15}^{11}, \qquad n_{11}^{11} = 396 - 4n_{14}^{11} - 15n_{15}^{11},$$

$$n_{12}^{11} = -550 + 6n_{14}^{11} + 20n_{15}^{11}, \qquad n_{13}^{11} = 330 - 4n_{14}^{11} - 10n_{15}^{11}.$$

Consider now, for example, $n_{15}^{11} = 11$. By the last equality, $n_{14}^{11} \leqq 55$, and by the third equality, $n_{14}^{11} \geqq 55$. Hence $n_{14}^{11} = 55$ and all other n's are uniquely determined.

LEMMA 4. *If an orthogonal array of* 12 *constraints exists, it must satisfy one of the two solutions*

Solution	n_{ij}					
	n_{15}^{12}	n_{14}^{12}	n_{13}^{12}	n_{12}^{12}	n_{11}^{12}	n_{10}^{12}
I′	27	42	2	0	3	6
II′	28	39	4	2	0	7

PROOF. Equations (1) imply for $k = 12$

$$3n_{13}^{12} + 2n_{12}^{12} = -264 + 10n_{15}^{12}, \qquad 3n_{11}^{12} + 2n_{12}^{12} = 144 - 5n_{15}^{12}.$$

Thus $27 \leqq n_{15}^{12} \leqq 28$. The same method as before shows that I′ and II′ are the only solutions of equations (1) for $k = 12$ and $n_{15}^{12} = 27$ or $n_{15}^{12} = 28$.

THEOREM 2. *The number of constraints of an orthogonal array does not exceed* 11, *provided that* $\lambda = s = t = 3$.

PROOF. It is enough to prove that neither Solution I′ nor II′ can be satisfied by an orthogonal array. As to solution I′, since $n_{10}^{12} = 6$, every 11-rowed subarray would have to satisfy the inequality $n_{10}^{11} \geqq 6$. Hence it remains to consider solutions IV, VI, VII, VIII and IX. It is easy to rule out all except VII since the rest could lead to 12-rowed arrays for which, if $n_{10}^{12} = 6$, n_{11}^{12} could be at most equal to two, contrary to the assumption that it is three. Hence, every 11-rowed subarray of an array satisfying I′ would have to satisfy VII. This is impossible, since $n_{11}^{12} \neq 0$. Thus, if we delete from the array a row which contains a zero belonging to a column having just one zero, we must get an 11-rowed subarray for which $n_{10}^{11} \geqq 7$.

The proof regarding solution II′ is analogous. Here every 11-rowed subarray would have to satisfy either VIII or IX. Solution IX is impossible, since it could lead to a 12-rowed array for which, if $n_{12}^{10} = 7$, n_{12}^{12} is at most equal to one. As before, it is impossible that every 11-rowed subarray of an array satisfying II′ will satisfy IX, because deletion from this array of a row containing a zero belonging to a column which has two zeros only must yield an 11-rowed subarray for which $n_{11}^{11} > 0$. This completes the proof of Theorem 2.

It is easy to show that no 11-rowed orthogonal arrays exist which satisfy solutions II, III, or V. As to the remaining solutions, the problem is unsolved.

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REFERENCES

[1] R. C. BOSE AND K. A. BUSH, "Orthogonal arrays of strengths two and three," *Ann. Math. Stat.*, Vol. 23 (1952), pp. 508–524.