CHARACTERIZATIONS OF COMPLETE CLASSES OF TESTS OF SOME MULTIPARAMETRIC HYPOTHESES, WITH APPLICATIONS TO LIKELIHOOD RATIO TESTS[1]

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1. Summary. For the problem of testing a simple hypothesis on a density function of the form $f_\theta(e) = \exp\{\psi_0(\theta) + \sum_1^k \psi_i(\theta)t_i(e) + t_0(e)\}$, explicit characterizations are given of a minimal essentially complete class of tests, the minimal complete class, and the closure of the class of Bayes' solutions, under certain assumptions. Applications are made to discrete distributions of the above form and to some problems of testing composite hypotheses. The likelihood ratio tests of these hypotheses are characterized and shown to be admissible under certain assumptions.

2. Introduction. Consider a probability density function of the form

$$g_\psi(e) = \exp\left\{\psi_0 + \sum_1^k \psi_i t_i(e) + t_0(e)\right\},$$

with e a sample point in an n-dimensional Euclidean space, and $\psi = (\psi_1, \cdots, \psi_k)$ a parameter in a subset Ψ of a k-dimensional Euclidean space. It can be shown [1] that $t = (t_1, \cdots, t_k) = [t_1(e), \cdots, t_k(e)]$ is a sufficient statistic which admits a density

$$p_\psi(t) = \exp\left\{\psi_0 + \sum_1^k \psi_i t_i + t_0\right\}$$

with respect to a measure in a k-dimensional Euclidean space, and that the family of such densities with $\psi \varepsilon \Psi$ is strongly complete. A family M of measures μ_θ on T is called *strongly complete* [1] if $\int_T f(t)d\mu_\theta = 0$ a.e. in Lebesgue measure on Ψ implies $f(t) = 0$ a.e. M.

We shall assume that $p_\psi(t)$ is a density with respect to Lebesgue measure unless the contrary is specified. Let T denote the set of values of t for which $p_\psi(t) > 0$ for some fixed $\psi \varepsilon \Psi$; we assume here that T does not vary with ψ.

We shall consider the problem of testing a hypothesis $H_0 : \psi \varepsilon \omega^0$ against an alternative $H_1 : \psi \varepsilon \omega'$, where ω^0 and ω' are disjoint subsets of Ψ. For the one-parameter case ($k = 1$), complete class results have been obtained by Lehmann [2] and by Blackwell, Girshick, and Rubin (cf. [3]), and certain minimax results have been obtained by Allen [4]. The present paper contains generalizations of Lehmann's results. We shall be concerned in the following with the case in which ω^0 consists of a point ψ^0 and $\omega' = \Psi - \omega^0$, except where otherwise specified.

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For all statistical purposes we can, as shown in [5], restrict our consideration to tests of H_0 each characterized by a measurable decision function $\delta = \delta(t)$, where $0 \leqq \delta(t) \leqq 1$, defined on T, such that when t is observed the test rejects H_0 with probability $\delta(t)$.

For any test δ, let $\beta_\delta(\psi)$ denote the probability that the test will accept H_0 when ψ is the parameter point; that is $\beta_\delta(\psi) = 1 - \int_T p_\psi \delta(t)\, dt$. We define the risk function $r_\delta(\psi)$ of a test δ as follows:

$$r_\delta(\psi) = \begin{cases} \beta_\delta(\psi) & \psi \,\varepsilon\, \omega', \\ 1 - \beta_\delta(\psi) & \psi \,\varepsilon\, \omega^0, \\ 0 & \psi \,\varepsilon\, (\Psi - \omega^0 - \omega'). \end{cases}$$

Let $\xi(\psi)$ denote a cumulative distribution function defined on Ψ. A Bayes' solution with respect to $\xi(\psi)$ is a test δ_ξ such that the "Bayes' risk" $r_\delta(\xi) = \int_\Psi r_\delta(\psi)\, d\xi(\psi)$ is minimized by taking $\delta = \delta_\xi$. Let B denote the set of all Bayes' solutions; that is $\delta \,\varepsilon\, B$ if and only if there exists a ξ such that δ is a Bayes' solution with respect to ξ. Let V' denote the set of all tests δ such that δ is non-randomized (i.e., $\delta(t)$ is 0 or 1 only) and the set A on which $\delta(t) = 0$ (the test's acceptance region) is the common part of T and some open convex set in k-dimensional Euclidean space. Let V denote the set of all tests δ such that $\delta \,\varepsilon\, V$ if and only if there exists a $\delta' \,\varepsilon\, V'$ such that the set on which $\delta'(t) \neq \delta(t)$ has Lebesgue measure zero.

The following are the usual definitions of some terms which will be used below: A class of tests is called *complete* if for every test δ outside the class there is a test δ' in the class which is uniformly better, that is such that $r_{\delta'}(\psi) \leqq r_\delta(\psi)$ for all ψ and $r_{\delta'}(\psi) < r_\delta(\psi)$ for some ψ. A class of tests is called *minimal complete* if it is complete but has no proper subset which is complete. A class of tests is called *essentially complete* if for every test δ outside the class there is a test δ' in the class such that $r_{\delta'}(\psi) \leqq r_\delta(\psi)$ for all ψ. A test is called *admissible* if there exists no uniformly better test. A class of tests is called *minimal essentially complete* if it is essentially complete but has no proper subclass with this property. A test δ will be called *uniformly as good* as a test δ' if $r_\delta(\psi) \leqq r_{\delta'}(\psi)$ for all ψ.

3. Characterizations of complete and essentially complete classes. The following theorem has been proved by Reiersøl [7].

THEOREM 1. *B is a subset of V.*

PROOF. For any ξ, let γ be the saltus of ξ at $\psi = \psi^0$. Then

$$\begin{aligned} r_\delta(\xi) &= \int_\Psi r_\delta(\psi)\, d\xi(\psi) = \int_\Psi \beta_\delta(\psi)\, d\xi(\psi) + \gamma(1 - 2\beta_\delta(\psi^0)) \\ &= \int_\Psi \left[\int_T \{1 - \delta(t)\} p_\psi(t)\, dt\right] d\xi(\psi) + \gamma - 2\gamma \int_T \{1 - \delta(t)\} p_{\psi^0}(t)\, dt \\ &= \gamma + \int_T \left[\left(\int_\Psi p_\psi(t)\, d\xi(\psi)\right) - 2\gamma p_{\psi^0}(t)\right] \{1 - \delta(t)\}\, dt. \end{aligned}$$

Let $Q(t) = \int_\Psi p_\psi(t)\, d\xi(\psi) - 2\gamma p_{\psi^0}(t)$, so that

$$r_\delta(\xi) = \int_T Q(t)(1 - \delta(t))\, dt + \gamma.$$

Clearly $r_\delta(\xi)$ is minimized by $\delta = \delta_\xi$ such that

$$\delta_\xi(t) = \begin{cases} 0 & t \,\varepsilon\, A_\xi = \{t \mid Q(t) < 0\}, \\ 1 & \text{all other } t. \end{cases}$$

But $Q(t) < 0$ is equivalent to

$$\begin{aligned} 2\gamma &> \int_\Psi \left(\frac{p_\psi(t)}{p_{\psi^0}(t)}\right) d\xi(\psi) \\ &= \int_\Psi \exp\left\{\sum_{i=1}^k (\psi_i - \psi_i^0)t_i + (\psi_0 - \psi_0^0)\right\} d\xi(\psi) \\ &= R(t), \text{ say.} \end{aligned}$$

Thus $A_\xi = \{t \mid R(t) < 2\gamma, \quad t \,\varepsilon\, T\}$.

Now for $0 < \lambda < 1$ and for any real u and v, we have

$$e^{\lambda u + (1-\lambda)v} \leqq \lambda e^u + (1 - \lambda)e^v,$$

with strict inequality holding in case $u \neq v$. Thus if t' and t'' are distinct points in A_ξ, then for $0 < \lambda < 1$ we have

$$R(\lambda t' + [1 - \lambda]t'') < \lambda R(t') + (1 - \lambda)R(t'') < 2\gamma,$$

proving that $\delta_\xi \,\varepsilon V'$. Furthermore $R(t) = 2\gamma$ holds only on a set of Lebesgue measure zero. For

$$\frac{\partial^2 R(t)}{\partial t_j^2} = \int_\Psi (\psi_j - \psi_j^0)^2 \exp\left\{\sum_1^k (\psi_i - \psi_i^0)t_i + (\psi_0 - \psi_0^0)\right\} d\xi(\psi)$$

is nonnegative for all t, for each j. Moreover, if for any j and t, $\partial^2 R(t)/\partial t_j^2 = 0$, then $\Pr\{\psi_j - \psi_j^0 \mid \xi\} = 1$. If for some t we have $\partial^2 R(t)/\partial t_j^2 = 0$ for $j = 1, \cdots k$, then $\Pr\{\psi = \psi^0 \mid \xi\} = 1$.

In the latter case the Bayes' acceptance regions are T itself and all subsets which differ from T by sets of Lebesgue measure zero. In the remaining case we have that for each $t \,\varepsilon\, T$ there exists a j for which $\partial^2 R(t)/\partial t_j^2 > 0$. Hence $R(t) = 2\gamma$ holds at most on the boundary points of the convex acceptance region A_ξ, which constitute a set of measure zero.

Since $R(t) = 2\gamma$ holds only on a set of Lebesgue measure zero, every Bayes' solution with respect to ξ must differ from δ_ξ at most on a set of Lebesgue measure zero. Hence every Bayes' solution is in V. Q.E.D.

Wald ([6], Theorem 5.8) has proved that under assumptions which are satisfied by $p_\psi(t)$, an essentially complete class of tests is constituted by the closure

$\bar{B}$ of B in the sense of regular convergence, which is defined as follows: $\lim_{i\to\infty} \delta_i = \delta_0$ in the regular sense if

$$\lim_{i\to\infty} \int_R \delta_i(t)\, dt = \int_R \delta_0(t)\, dt$$

for any bounded subset R of the sample space T. If $\lim_{i\to\infty} \delta_i = \delta_0$ except on a set of Lebesgue measure zero, then it is clear that δ_i converges to δ_0 also in the regular sense. For the special case of $\delta_i \varepsilon V$ (but not in general) the following converse is also true.

Lemma. *Let* $\lim_{i\to\infty} \delta_i = \delta_0$ *in the regular sense, where* $\delta_i(t) = 0$ *on a convex set* A_i, *and* $\delta_i(t) = 1$ *elsewhere. Then* $\lim_{i\to\infty} \delta_i = \delta_0$ *except on a set of Lebesgue measure zero. Furthermore* $\delta_0 \varepsilon V$, *and except on a set of Lebesgue measure zero,*

$$\delta_0(t) = \begin{cases} 0 & \text{on } a_0 = \lim_{i\to\infty} \bigcap_{j \ge i} A_j, \\ 1 & \text{elsewhere.} \end{cases}$$

Proof. Let

$$A_0 = \lim_{i\to\infty} \bigcup_{j\ge i} A_j, \qquad a_0 = \lim_{i\to\infty} \bigcap_{j\ge i} A_j, \qquad D = A_0 - a_0 .$$

It is clear that a_0 is convex. If $\lim_{i\to\infty} \delta_i(t)$ does not exist almost everywhere in Lebesgue measure, then $\mu(D) > 0$, where μ denotes Lebesgue measure. For infinitely many A_i we have $D \subset A_i$, and for infinitely many A_i we have $DA_i = 0$.

Let h be a point of D which is not a boundary point of a_0. Let H be the smallest convex set containing a_0 and h, and let k be an interior point of $H - a_0$. Such points h and k exist except in the trivial cases $\mu(D) = 0$ or $\mu(a_0) = 0$.

Let K be the convex cone consisting of the half-lines from k through the points of a_0. Let K^* be the "negative" cone consisting of the other halves of the same lines. Then $\mu(HK^*) > 0$. Since $h \varepsilon D$, we have $HK^* \subset A_i$ for infinitely many A_i, and $HK^*A_i = 0$ for infinitely many A_i. Let R be a bounded subset of HK^* with $\mu(R) > 0$. Then $\int_R \delta_i(t)\, dt = \mu(R) > 0$ for infinitely many i, and also $\int_R \delta_i(t)\, dt = 0$ for infinitely many i. Thus δ_i do not converge in the regular sense if $\mu(D) > 0$, proving the first assertion of the lemma.

If the δ_i converge in the regular sense, and hence $\mu(D) = 0$, the second assertion of the lemma follows from the fact that a_0 is convex.

From the lemma it follows that $\bar{V} = V$, where $\bar{V}$ denotes the closure of V in the regular sense. Since $B \subset V$ by Theorem 1, $\bar{B} \subset \bar{V} = V$. Since $\bar{B}$ is essentially complete we have

Corollary 1. *V is an essentially complete class.*

Since for every test in V there is a test in V' with identical risk function, we have

Corollary 2. *V' is an essentially complete class.*

The above-mentioned completeness property of the family of distributions $p_\psi(t)$ implies that two decision functions with identical risk functions differ at most on a subset of T with Lebesgue measure 0. Hence we have

COROLLARY 3. *V is a complete class.*

In the following sections it will be shown that, under certain additional assumptions, affirmative answers to the following questions hold.

1) Does $\bar{B}$ coincide with V?
2) Is V the minimal complete class?
3) Is V' a minimal essentially complete class?

In general V will not be minimal complete if ω' is a proper subset of $\Psi - \omega^0$. For example, consider the case in which $p_\psi(t) = (2\pi)^{-1/2} \exp \{-\frac{1}{2}(t_1 - \psi_1)^2\}$, with ω^0 and ω' consisting respectively of $\psi_1 = 0$ and $\psi_1 = 1$, and with $k = 1$. Here the minimal complete class is known to consist of the class of best tests of all sizes. The acceptance regions of these tests are all the intervals which are infinite to the left and all the sets which coincide with such intervals to within sets of Lebesgue measure zero. Since these acceptance regions are a proper subclass of V, V is not minimal complete.

However, V is known to be minimal complete in some one-parameter cases. If we take the preceding example with ω' altered to consist of $\psi_1 = 1$ and $\psi_1 = -1$, the minimal complete class consists of all "two-tail" tests of all sizes ([6], pp. 136–138). The acceptance regions of these tests are all the intervals and all the sets differing from intervals by sets of Lebesgue measure zero, that is just the class V. Here V', the class of all open intervals, is clearly minimal essentially complete.

For other density functions $p_\psi(t)$ with $k = 1$ (that is, one-parameter distributions) the same properties can be proved for V and V' provided ω' contains a point $\psi_1' < \psi_1^0$ and a point $\psi_1'' > \psi_1^0$ (cf. [2]). While the following two sections give the same conclusions for $k > 1$, they necessarily require stronger restrictions on ω'. Thus the one-parameter results referred to will not all be contained as special cases.

4. Full characterization of $\bar{B}$. Hereafter "a test A" will mean a (nonrandomized) test δ with $\delta = 0$ on A and $\delta = 1$ on $T - A$.

ASSUMPTION 1. The subset ω' contains hyperspheres of arbitrarily large radii. That is, for some increasing unbounded sequence $d_1, d_2, \cdots$, all formal solutions ψ of the equations $d_j^2 = \sum_1^k (\psi_i - \psi_i^0)^2$ for $j = 1, 2, \cdots$ are contained in ω'. This assumption is satisfied if Ψ is k-dimensional Euclidean space and $\Psi - \omega'$ is bounded.

THEOREM 2. *If Assumption* 1 *is satisfied*, $V = \bar{B}$.

PROOF. Since $V = \bar{V}$, and by Theorem 1, $B \subset V$, we have $\bar{B} \subset V$. It remains to show that $V \subset \bar{B}$. The method of proof will be to show that every $A \in V$ is an essentially unique limit (in the regular sense) of a sequence of Bayes' solutions $\{A_{\xi_i}\}$.

LEMMA. *For every bounded convex set C, the test $A = CT$ is an essentially unique limit, in the regular sense, of a sequence of Bayes' solutions.*

Proof of Lemma: Let A be the set-theoretic product of T and any bounded convex set C in k-dimensional Euclidean space. For each point $\psi \,\varepsilon\, (\Psi - \omega^0)$, let $\eta(\psi)$ be the quantity such that

$$\sum_{i=1}^{k} \left[\frac{\psi_i - \psi_i^0}{d(\psi)}\right] t_i + \eta(\psi) = 0, \qquad d(\psi) = + \sqrt{\sum_{i=1}^{k} (\psi_i - \psi_i^0)^2},$$

is the equation of a supporting hyperplane of C, and such that the left member of this equation is negative for t any interior point of C. For any $d > 0$, let

$$\omega_d = \{\psi \mid d(\psi) = d\}, \qquad \rho(\psi) = \exp\{(\psi_0^0 - \psi_0) + \eta(\psi)\, d(\psi)\}$$

Let $\sigma_d = \int_{\omega_d} d\omega_d$, where $d\omega_d$ denotes an infinitesimal element of $(k - 1)$-dimensional Lebesgue measure on ω_d. Let

$$\frac{1}{\gamma_d} = 1 + \frac{1}{\sigma_d} \int_{\omega_d} \rho(\psi)\, d_{\omega_d}.$$

Then we define the cumulative distribution functions $\xi_d = \xi_d(\psi)$ as follows: ξ_d has a saltus γ_d at $\psi = \psi^0$, and for any subset ν of Ψ-ω^0,

$$\int_{\nu} d\xi_d(\psi) = \frac{\gamma_d}{\sigma_d} \int_{\nu\omega_d} \rho(\psi)\, d\omega_d.$$

Consider the Bayes' solution with respect to ξ_d given by

$$A_d = \{t \mid R_d(t) < 2\gamma_d, \quad t \,\varepsilon\, T\}$$

where, as in the proof of Theorem 1,

$$R_d(t) = \int_{\Psi} \exp\left\{\sum_{i=1}^{k} (\psi_i - \psi_i^0) t_i + (\psi_0 - \psi_0^0)\right\} d\xi_d(\psi).$$

Here

$$R_d(t) = \gamma_d + \frac{\gamma_d}{\sigma_d} \int_{\omega_d} \exp\left\{\sum_{i=1}^{k} (\psi_i - \psi_i^0) t_i + \eta(\psi)\, d(\psi)\right\} d\omega_d.$$

Thus

$$A_d = \left\{t \,\middle|\, 1 > \frac{1}{\sigma_d} \int_{\omega_d} \exp\left(d \cdot \left[\sum_{i=1}^{k} \frac{(\psi_i - \psi_i^0)}{d} t_i + \eta(\psi)\right]\right) d\omega_d,\ t \,\varepsilon\, T\right\}.$$

Now for t any interior point of A and $d > 0$, the exponent in the integrand in the preceding expression is d multiplied by a negative quantity which is numerically never less than $\Delta(t)$, for $\psi \,\varepsilon\, \omega_d$, where $\Delta(t)$ is the Euclidean distance from t to the closest boundary point of A. Thus as d increases the integrand approaches 0 uniformly over ω_d, and the inequality is satisfied for all sufficiently large values of d. Hence each interior point t of A is an interior point of A_d for all d exceeding some d_t.

For every exterior point t not a boundary point of A, let ω_d^t denote the subset of ω_d such that for each $\psi \,\varepsilon\, \omega_d^t$, the equation

$$\sum_{i=1}^{k} \left(\frac{\psi_i - \psi_i^0}{d(\psi)}\right) t_i + \eta(\psi) = 0$$

determines a plane which separates t from A; that is the left member of the preceding equation is positive for all $\psi \varepsilon \omega_d^t$. Since $\eta(\psi)$ is continuous, $\int_{\omega_d^t} d\omega_d = m(\omega_d^t)$, say, is positive. We can choose $\epsilon > 0$ such that for each $d > 0$, the subset ω_d^ϵ of ω_d^t such that

$$\sum_{i=1}^{k} \left(\frac{\psi_i - \psi_i^0}{d(\psi)}\right) t_i + \eta(\psi) \geqq \epsilon, \qquad \psi \varepsilon \omega_d^\epsilon,$$

satisfies $\int_{\omega_d^\epsilon} d\omega_d \geqq \frac{1}{2}m(\omega_d^t)$. Hence in the inequality which defines A_d above, the exponent in the integrand is not less than $d \cdot \epsilon$ on a set of Lebesgue measure not less than $\frac{1}{2}m(\omega_d^t)$. Hence as d increases, the right member of the inequality approaches ∞, and the exterior point t of A is also an exterior point of A_d for all d exceeding some d_t'. Hence we have that A differs from $\lim_{d \to \infty} A_d$ at most by a subset of the boundary of A, which has Lebesgue measure zero, proving the lemma.

PROOF OF THEOREM 2. Since every unbounded convex set is an essentially unique limit, in the regular sense, of a sequence of bounded convex sets, and hence is included in the closure of any class which contains all bounded convex sets, Theorem 2 is proved.

EXAMPLE 1. Let $p_\psi(t) = (1/2\pi) \exp[-\frac{1}{2}\{(t_1 - \psi_1)^2 + (t_2 - \psi_2)^2\}]$. With Ψ as the (ψ_1, ψ_2) plane and $\omega^0 = \{(\psi_1^0, \psi_2^0)\}$, the subset $\omega' = \Psi - \omega^0$ contains all solutions (ψ_1, ψ_2) of $d^2 = (\psi_1 - \psi_1^0)^2 + (\psi_2 - \psi_2^0)^2$ for $d > 0$. Hence the convex sets in the (t_1, t_2) plane are an essentially complete class for testing H_0 : $(\psi_1, \psi_2) = (\psi_1^0, \psi_2^0)$. A similar conclusion will clearly hold when $p_\psi(t)$ is any multivariate normal distribution with known covariance matrix and unknown mean ψ.

5. Characterization of a minimal essentially complete class.

CASE 1: *Bounded acceptance regions.*

ASSUMPTION 2. t_0 is a continuous function of t.

Let $\mathfrak{D}_b$ denote the class of tests A of H_0 whose acceptance regions are bounded. Let V_b' denote the class of tests in V' with bounded acceptance regions.

THEOREM 3. *If A' and A'' are distinct tests in V_b', then neither test is uniformly as good as the other, if Ψ satisfies Assumptions* 1 *and* 2.

COROLLARY 1. *If T is bounded, and if Assumptions* 1 *and* 2 *are satisfied, then V' is a minimal essentially complete class.*

PROOF OF COROLLARY 1. When T is bounded, V_b' coincides with V' which is, by Corollary 2 of Theorem 1, an essentially complete class. But an essentially complete class, no member of which is uniformly as good as any other, is minimal essentially complete.

PROOF OF THEOREM 3. Let A', A'' be distinct tests in V_b'. Assume that

$$r_{A'}(\psi^0) = r_{A''}(\psi^0).$$

Then each of the sets $A' - A''$ and $A'' - A'$ has positive Lebesgue measure. Let t^* be any interior point of $A' - A''$, and let T' be any supporting hyperplane of A'' which separates t^* form A''.

Let the equation of T' be written $\sum_1^k a_i t_i + a_0 = 0$, with $\sum_1^k a_i^2 = 1$. Let $\Delta(t) = \sum_1^k a_i t_i + a_0$; that is, $\Delta(t)$ is the directed distance from T' to any point t. Let

$$A^* = \left\{ t \,\middle|\, \sum_1^k (t_i - t_i^*)^2 \leqq \tfrac{1}{4} \left|\Delta(t^*)\right|^2, \quad t \,\varepsilon\, A' \right\}.$$

Then A^* has measure $L(A^*) > 0$. For t' any point in A^*, and t'' any point of $\overline{A''}$, the closure of A'', we have

$$|\Delta(t') - \Delta(t'')| \geqq \tfrac{1}{2}|\Delta(t^*)| > 0.$$

Let $\psi^c = c(a_1, \cdots a_k)$ for any real c, and let

$$g(t', t'', c) = \log [p_{\psi^c}(t')/p_{\psi^c}(t'')] - (t_0' - t_0'') = c \sum a_i(t_i' - t_i'')$$
$$= \pm c \,|\Delta(t') - \Delta(t'')|.$$

Now by Assumption 1, ω' contains points ψ^c with $c \to \infty$ and with $c \to -\infty$. Hence we can choose a sequence $\psi^{c_1}, \psi^{c_2}, \cdots$ such that $\lim_{i\to\infty} |c_i| = \infty$ and all c_i are of the same sign as $\sum_1^k a_i(t_i' - t_i'')$, for $t' \,\varepsilon\, A^*$ and $t'' \,\varepsilon\, A''$. By Assumption 3, $|t_0' - t_0''|$ is continuous and hence bounded say by v, for $t' \,\varepsilon\, A^*$ and $t'' \,\varepsilon\, \overline{A''}$. But

$$\log[p_\psi c_i(t')/p_\psi c_i(t'')] = \pm c_i \,|\, \Delta(t') - \Delta(t'') \,|\, + (t_0' - t_0'')$$
$$> \tfrac{1}{2}\, |c_i| \cdot |\Delta(t^*)| - v.$$

Hence $\lim_{i\to\infty} [p_\psi c_i(t')/p_\psi c_i(t'')] = \infty$ uniformly for $t' \,\varepsilon\, A^*$ and $t'' \,\varepsilon\, \overline{A''}$. That is, for any $M > 0$ there exists a c' such that $p_\psi c'(t')/p_\psi c'(t'') > M$ for all $t' \,\varepsilon\, A^*$ and $t'' \,\varepsilon\, \overline{A''}$. Let

$$\underline{p} = \inf_{t' \varepsilon A^*} p_\psi c'(t'), \qquad \bar{p} = \sup_{t'' \varepsilon A''} p_\psi c'(t'');$$

then $\underline{p}/\bar{p} \geqq M$. Now

$$r_{A'}(\psi^{c'}) \geqq \int_{A^*} p_\psi c'(t)\, dt \geqq L(A^*)\underline{p},$$

$$r_{A''}(\psi^{c'}) = \int_{A''} p_\psi c'(t)\, dt \leqq L(A'')\bar{p},$$

where $L(A'')$ is the measure of A''. Hence

$$\frac{r_{A'}(\psi^{c'})}{r_{A''}(\psi^{c'})} \geqq \frac{L(A^*)\cdot \underline{p}}{L(A'')\cdot \bar{p}} \geqq \frac{L(A^*)}{L(A'')} \cdot M.$$

But M may be chosen arbitrarily large, in particular so large that

$$M > L(A'')/L(A^*).$$

In this case we obtain $r_{A'}(\psi) > r_{A''}(\psi)$ for $\psi =$ some ψ^{c_i}

Therefore A' is not uniformly as good as A''. Interchanging A' and A'' in the preceding discussion, we conclude also that A'' is not uniformly as good as A'.

It remains to consider tests A' and A'' in V_b' such that

$$r_{A'}(\psi^0) \neq r_{A''}(\psi^0).$$

Assume $r_{A'}(\psi^0) < r_{A''}(\psi^0)$. Then $A' - A''$ has positive measure. The proof given above for the case $r_{A'}(\psi^0) = r_{A''}(\psi^0)$ then shows that there exists a $\psi \varepsilon \Psi$ such that $r_{A'}(\psi) > r_{A''}(\psi)$. Similarly if $r_{A'}(\psi^0) > r_{A''}(\psi^0)$, we can show that there exists a $\psi \varepsilon \Psi$ such that $r_{A'}(\psi) < r_{A''}(\psi)$. Q.E.D.

The method of proof of Theorem 3 can be applied directly to prove that no test not in V is uniformly as good as *any* test in V_b'. Hence we have

COROLLARY 2. *If Assumptions* 1 *and* 2 *are satisfied and T is bounded, then V is the minimal complete class.*

These results are used in connection with Example 1 in section 10, below.

6. Characterization of a minimal essentially complete class.

CASE 2. *The General Case.* ASSUMPTION 3. Let T' be the common part of T and any half-space, that is, $T' = \{t \mid \sum a_i t_i + a_0 \leqq 0,\ t \varepsilon T\}$, for arbitrary a_i's subject to $\sum a_i^2 = 1$. Let t' be any point in $T - T'$. Then there exists a sequence of points ψ^i in ω' such that, uniformly in some neighborhood of t',

$$\lim_{i\to\infty} \left[p_{\psi^i}(t') \Big/ \int_{T'} p_{\psi^i}(t)\, dt \right] = \infty.$$

THEOREM 4. *If Assumptions* 2 *and* 3 *are satisfied, V' is a minimal essentially complete class.*

PROOF. By Corollary 2 of Theorem 1, V' is essentially complete. It remains to show that, of any pair of distinct tests A' and A'' in V', neither test is uniformly as good as the other.

Given any such pair of tests, at least one of the sets $A' - A''$ and $A'' - A'$ has positive measure. Assume $A' - A''$ has positive measure, and let t^* be any interior point of $A' - A''$. Let $\Delta(t) = \sum a_i t_i + a_0$. Let

$$A^* = \left\{ t \,\middle|\, \sum_{i=1}^{k} (t_i - t_i^*)^2 \leqq |\Delta(t^*)|^2 \epsilon^2, \quad t \varepsilon A' \right\}.$$

Then for some ϵ, with $0 < \epsilon < 1$, we have by Assumption 3 that for arbitrarily large M, there exists i_M such that for all $t' \varepsilon A_\epsilon^*$,

$$p_{\psi^i}(t') \Big/ \int_{T'} p_{\psi^i}(t)\, dt > M, \qquad i \geqq i_M.$$

Let $p = \inf_{t \varepsilon A_\epsilon^*} p_{\psi^{i_M}}(t)$. Let $L(A_\epsilon^*)$ be the measure of A_ϵ^*. Then

$$p \geqq M \cdot \int_{T'} p_{\psi^{i_M}}(t)\, dt.$$

Now

$$r_{A'}(\psi^{iM}) \geqq \int_{A_\epsilon^*} p_{\psi^{iM}}(t)\,dt \geqq L(A_\epsilon^*)\cdot p,$$

$$r_{A''}(\psi^{iM}) \leqq \int_{T'} p_{\psi^{iM}}(t)\,dt.$$

Hence $r_{A'}(\psi^{iM}) \geqq M\cdot L(A_\epsilon^*)\cdot r_{A''}(\psi^{iM})$. But M may be chosen arbitrarily large, in particular so large that $M > 1/L(A_\epsilon^*)$, in which case we obtain

$$r_{A'}(\psi^{iM}) > r_{A''}(\psi^{iM}).$$

Thus A' is not uniformly as good as A''.

If $r_{A'}(\psi^0) = r_{A''}(\psi^0)$, we have that $A'' - A'$, as well as $A' - A''$, has positive measure. By interchanging A' and A'' in the preceding discussion we conclude also that A'' is not uniformly as good as A'. If $r_{A'}(\psi^0) < r_{A''}(\psi^0)$, then $A' - A''$ has positive measure and as above we conclude that $r_{A'}(\psi) > r_{A''}(\psi)$ for some ψ. Similarly if $r_{A'}(\psi^0) > r_{A''}(\psi^0)$, we conclude as above that $r_{A'}(\psi) < r_{A''}(\psi)$ for some ψ. Q.E.D.

The method of proof of Theorem 4 can be applied directly to prove that no test not in V is uniformly as good as *any* test in V'. Hence we have

COROLLARY. *If Assumptions* 2 *and* 3 *are satisfied, then* V *is the minimal complete class.*

EXAMPLE. Let $p_\psi(t) = (1/2\pi)\exp\{-\sum_1^2 \frac{1}{2}(x_i - \mu_i)^2\}$, and let H_0 specify $(\mu_1, \mu_2) = (0, 0)$. For any a_1, a_2, and a_0 such that $a_1^2 + a_2^2 = 1$, let

$$T' = \{(x_1, x_2) \mid a_1x_1 + a_2x_2 + a_0 \leqq 0\},$$

$$y = y(t) = (y_1, y_2) = (a_1x_1 + a_2x_2, -a_2x_1 + a_1x_2).$$

Then we may write $p_\psi(t) = (1/2\pi)\exp\{-\frac{1}{2}\sum_1^2 (y_i - \nu_i)^2\}$, with $H_0 : (\nu_1, \nu_2) = (0, 0)$, and $T' = \{(y_1, y_2) \mid y_1 \leqq -a_0\}$. Let $t' = (y_1, y_2)$ be any point not in T'; that is, $y_1' > -a_0$. Let

$$v = \frac{p_\psi(t')}{\int_{T'} p_\psi(t)\,dt}$$

$$= \frac{(1/2\pi)\exp\{-\frac{1}{2}[(y_1' - \nu_1)^2 + (y_2' - \nu_2)^2]\}}{(1/2\pi)\int_{-\infty}^{\infty}\left[\int_{-\infty}^{-a_0}\exp\{-\frac{1}{2}(y_1 - \nu_1)^2\}\,dy_1\right]\exp\{-\frac{1}{2}(y_2 - \nu_2)^2\}\,dy_2}.$$

Proceeding as in [8], we set

$$\nu_2 = 0, \qquad u = -a_0 - \nu_1, \qquad \Delta = y_1' + a_0, \qquad h = (2\pi)^{-1/2}\exp(-\tfrac{1}{2}y_2'^2).$$

Then $v = h\exp[-\frac{1}{2}(u + \Delta)^2] \Big/ \int_{-\infty}^{u} \exp[-\frac{1}{2}r^2]\,dr$.

Now for $\nu_1 > y_1'$,

$$u \equiv -a_0 - \nu_1 < u + \Delta < 0, \qquad u^2 > (u + \Delta)^2,$$

so that $\exp[-\frac{1}{2}(u + \Delta)^2] > \exp[-\frac{1}{2}u^2]$.

Similarly for $\nu_2 = 0$ and $\nu_1 > y_1'$,

$$v > h\cdot(\nu_1 + a_0), \qquad \lim_{\nu_1=\infty} v = \infty.$$

Thus Assumption 3 is satisfied, and by Theorem 4 we conclude that the open convex sets in the (x_1, x_2) plane constitute a minimal essentially complete class of acceptance regions (tests) of H_0.

7. Other cases. While Theorem 1 shows that $\bar{B} \subset V$ holds without restrictions on ω', an example above showed that even with $k = 1$ we do not have $\bar{B} = V$ unless ω' is sufficiently inclusive. For $k > 1$, when ω' does not satisfy Assumption 1 it is possible in some cases to use the methods of the preceding sections to obtain characterizations of minimal complete classes or at least characterizations of smaller complete classes than V.

Such possibilities will be illustrated in a special case in terms of Example 1 above. Let ω' now be

$$\{(\psi_1, \psi_2) \mid \psi_1 \geqq 0, \quad \psi_2 \geqq 0, \quad \psi_1 + \psi_2 > 0\};$$

that is ω' is the closed first quadrant with $\psi^0 = (0, 0)$ deleted. Then the method of Theorem 1 can be extended to show that every Bayes' acceptance region A_ξ has the property that if $(t_1, t_2) \; \varepsilon \; A_\xi$ and $t_1' \leqq t_1$, and $t_2' \leqq t_2$, then $(t_1', t_2') \; \varepsilon \; A_\xi$, with exceptions on sets of measure zero. The method of Theorem 3 can be adapted to prove that tests having the preceding property constitute the minimal complete class.

For cases in which ω' is bounded, in general no simple characterization of the minimal complete classes can be given. By extending the method of Theorem 1 it can be shown that when ω' is bounded the Bayes' acceptance regions A_ξ cannot approximate arbitrarily closely any bounded convex hyperpolygon. Thus a minimal essentially complete class consists of a proper subset of V' which can be described roughly as excluding all hyperpolygons with finite vertices and edges, and all acceptance regions which closely approximate such hyperpolygons; as the bounds on ω' are narrowed, more elements of V' are excluded.

For some common distributions of the form $p_\psi(t)$, Assumption 1 fails. Examples are $p_\psi(t)$ as determined by either

(a) $$g_\psi(e) = \frac{1}{(2\pi)^{N/2}\psi_2^N} \exp\left\{\frac{-1}{2\psi_2^2} \sum_{j=1}^{N} (x_i - \psi_1)^2\right\},$$

(b) $$g_\psi(e) = \frac{1}{(2\pi\psi_1\psi_2)^N} \exp\left\{\frac{-1}{2} \sum_{j=1}^{N} \left(\frac{x_{1j}^2}{\psi_1^2} + \frac{x_{2j}^2}{\psi_2^2}\right)\right\}.$$

For the problem of testing a simple hypothesis in (a), the density of a sample from a normal population with unknown mean and variance, Stein (in a private communication) has constructed a test which is in V, but which can be strictly improved on, showing that for this problem V is not a minimal complete class. For this testing problem, Walsh [9] has investigated the operating characteristics of three tests of H_0 as a basis for selecting tests for quality control and other applications. Only one of the three tests considered, that one whose acceptance region is a rectangle in the $(\sum x_i^2, \sum x_i)$-plane, is contained in V and hence may be admissible; the remaining two tests are inadmissible.

For the class of problems considered in this paper, no general method is yet known for constructing a test which is uniformly better than a given inadmissible test. On the other hand, when some possible tests for a given application are to be examined on the basis of their operating characteristics, it seems advisable to restrict consideration to the smallest known complete class of tests, except possibly where computational difficulties make this impractical.

8. Tests of composite hypotheses. The methods of proof of the theorems and corollaries above can be used to reach certain conclusions concerning tests of composite hypotheses on distributions of the form of $p_\psi(t)$. Let H_0 now specify $\psi \varepsilon \omega^0$, where ω^0 consists of two or more points.

From the proof of Theorem 1 we conclude that V contains at least all of the Bayes' acceptance regions for H_0 corresponding to ξ's such that $\int_{\omega^0} d\xi(\psi)$ is equal to the saltus of ξ at some point ψ in ω^0; that is, $V \cdot \bar{B} \neq 0$.

From the proof of Theorem 2 we conclude that, when Assumption 1 is satisfied with ψ^0 some point in ω^0, then $V \subset \bar{B}$.

For any δ, $r_\delta(\psi)$ is a continuous function of ψ, except at ψ^0. Suppose that δ_0 is admissible for testing $\{\psi^0\}$ against $\omega' = \Psi - \{\psi^0\}$. Let ω^* be a subset of ω' having k-dimensional Lebesgue measure zero. Then δ_0 remains admissible for testing $\{\psi^0\}$ against $\Psi - \{\psi^0\} - \omega^*$, and also for testing $\{\psi^0\} + \omega^*$ against $\psi - \{\psi^0\} - \omega^*$.

Theorem 3 is valid when H_0 is composite. The proof of Theorem 4 shows that when Assumptions 2 and 3 are satisfied, neither of any pair of distinct tests in V' is uniformly as good as the other; hence when H_0 is composite, any essentially complete class will contain V' or an equivalent class.

Lehmann [10] has defined a class of "monotone critical regions." As some examples in the following section will show, when Assumptions 2 and 3 are satisfied these tests are admissible and in some cases are also likelihood ratio tests.

9. Likelihood ratio tests. A likelihood ratio test of a (simple or composite) hypothesis $H_0 : \psi \varepsilon \omega^0$ against an alternative $H_1 : \psi \varepsilon \omega'$ has the following form: Reject H_0 if and only if $\lambda(t) < K$, where

$$\lambda(t) = \sup_{\psi \varepsilon \omega^0} p_\psi(t) \Big/ \sup_{\psi \varepsilon (\omega^0 + \omega')} p_\psi(t)$$

and K is a constant, $0 \leqq K \leqq 1$. Let $A_K(\omega^0, \omega^0 + \omega')$ denote the acceptance region of such a test. Then

$$\begin{aligned} A_K(\omega^0, \omega^0 + \omega') &= \{t \mid \sup_{\psi \varepsilon \omega^0} p_\psi(t) \geqq K \cdot \sup_{\psi \varepsilon (\omega^0 + \omega')} p_\psi(t), \quad t \varepsilon T\} \\ &= \bigcup_{\psi'' \varepsilon \omega^0} \{t \mid p_{\psi''}(t) \geqq K \cdot \sup_{\psi \varepsilon (\omega^0 + \omega')} p_\psi(t), \quad t \varepsilon T\} \\ &= \bigcup_{\psi'' \varepsilon \omega^0} \bigcap_{\psi' \varepsilon (\omega^0 + \omega')} \{t \mid p_{\psi''}(t) \geqq K \cdot p_{\psi'}(t), \quad t \varepsilon T\}. \end{aligned}$$

[These relations are independent of the form of $p_\psi(t)$. They generalize a remark of Neyman and Pearson ([11], p. 282). They provide a convenient construction of likelihood ratio tests in some cases in which the usual calculus methods fail or are cumbersome, as will be illustrated in an example below.]

Assume now that $p_\psi(t)$ has the form assumed above. If ω^0 consists of the point ψ^0 then

$$A_K(\omega^0, \omega^0 + \omega') = \bigcap_{\psi' \varepsilon (\omega^0 + \omega')} \{t \mid p_{\psi^0}(t) \geqq K \cdot p_{\psi'}(t), \quad t \,\varepsilon\, T\}$$

$$= \bigcap_{\psi' \varepsilon (\omega^0 + \omega')} \{t \mid \sum_{i=1}^{k} (\psi_i^0 - \psi_i')t_i \geqq \log K - (\psi_0^0 - \psi_0'), \quad t \,\varepsilon\, T\}.$$

Thus $A_K(\omega^0, \omega^0 + \omega')$ is a set-theoretic product of T and a set of "linear half-spaces" of T. Hence $A_K \,\varepsilon\, V$. Hence we conclude that the likelihood ratio tests of simple hypotheses are admissible if Assumption 1 or Assumption 2 holds.

For certain cases in which H_0 is composite it will again turn out that

$$A_K(\omega^0, \omega^0 + \omega') \,\varepsilon\, V,$$

so that the likelihood ratio test is admissible under the appropriate assumptions.

Example 1. Let $p_\psi(t) = (1/2\pi) \exp [-\frac{1}{2}\sum_1^2 (t_i - \psi_i)^2]$. If H_0 and H_1 are simple and $(\psi_1^0, \psi_2^0) = (0, 0)$, then

$$A_K(\omega^0, \omega^0 + \omega') = \left\{(t_1, t_2) \,\middle|\, \sum_{i=1}^{2} (-\psi_i' t_i) \geqq \log K - \tfrac{1}{2}(\psi_1'^2 + \psi_2'^2)\right\}.$$

This is a half-plane bounded by a line whose directed distance from the origin is $d = [\log K - \frac{1}{2}(\psi_1'^2 + \psi_2'^2)]/\sqrt{\psi_1'^2 + \psi_2'^2}$. If now we take

$$\omega' = \{(\psi_1, \psi_2) \mid \psi_1^2 + \psi_2^2 = \psi_1'^2 + \psi_2'^2\},$$

according to the relations above $A_K(\omega^0, \omega^0 + \omega')$ will be the common part of all half-planes whose boundaries have directed distance d from the origin; that is

$$A_K(\omega^0, \omega^0 + \omega') = \{(t_1, t_2) \mid t_1^2 + t_2^2 \leqq d\}.$$

If next we take ω' to be $\Psi - \omega^0$, where Ψ is the (ψ_1, ψ_2)-plane, $A_K(\omega^0, \omega^0 + \omega')$ will be the common part of a family of sets of the form $\{(t_1, t_2) \mid t_1^2 + t_2^2 \leqq d\}$ and so will itself be a set of this form. If finally we take ω^0 as any set in Ψ, with ω' its complement, we obtain that in this general case $A_K(\omega^0, \omega^0 + \omega') = A_K(\omega^0, \Psi)$, the set of points (t_1, t_2) whose distance from the set ω^0 does not exceed a certain constant c.

For example, if $\omega^0 = \{(\psi_1, \psi_2) \mid \text{one or both } \psi_i \leqq 0\}$, then $A_K(\omega^0, \Psi) = \{(t_1, t_2) \mid \text{one or both } t_i \leqq c\}$.

If $\omega^0 = \{(\psi_1, \psi_2) \mid \text{both } \psi_i \leqq 0\}$, then $A_K(\omega^0, \Psi) = \{(t_1, t_2) \mid (t_1 \leqq c, \quad t_2 \leqq 0)$, or $(t_1 \leqq 0, \quad t_2 \leqq c)$, or $(t_1 > 0, \quad t_2 > 0, \quad \sqrt{t_1^2 + t_2^2} \leqq c)\}$.

A sufficient (but not necessary) condition that $A_K(\omega^0, \Psi)$ be convex is that ω^0 be convex. If ω^0 is any convex set the methods of sections 7 and 8 show that the likelihood ratio test of H_0 is admissible.

EXAMPLE 2. Consider the problem of testing whether a given ranking of the means of k normal populations with known variances is correct. Let

$$p_\psi(t) = \frac{1}{(2\pi)^{k/2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{k}(t_i - \psi_i)^2\right].$$

Let H_0 specify that $\psi_1 \geqq \psi_2 \geqq \cdots \geqq \psi_k$. For simplicity, let

$$k = 3, \quad y_1 = (t_1 - t_2)/\sqrt{2}, \quad y_2 = (t_2 - t_3)/\sqrt{2}.$$

Then, as $y = (y_1, y_2)$ is a sufficient statistic for $\nu = (\psi_1 - \psi_2, \psi_2 - \psi_3)$, we may take

$$p_\nu(y) = (\tfrac{1}{2}\pi) \exp\left[-\tfrac{1}{2}[(y_1 - \nu_1)^2 - (y_1 - \nu_1)(y_2 - \nu_2) + (y_2 - \nu_2)^2]\right],$$

and write $H_0 : \nu_i \geqq 0$ for $i = 1, 2$. Using the expressions above, it is easily seen that $A_K(\omega^0, \Psi)$ is the set of points (y_1, y_2) swept out by an ellipse of the form

$$(y_1 - \nu_1)^2 - (y_1 - \nu_1)(y_2 - \nu_2) + (y_2 - \nu_2)^2 = c,$$

as its center (ν_1, ν_2) sweeps throughout the first quadrant; that is

$$\begin{aligned} A_K(\omega^0, \Psi) &= \{(y_1, y_2) \mid \text{for some } \nu_1 \geqq 0, \ \nu_2 \geqq 0, \\ &\qquad (y_1 - \nu_1)^2 - (y_1 - \nu_1)(y_2 - \nu_2) + (y_2 - \nu_2)^2 \leqq c\} \\ &= \{(y_1, y_2) \mid (y_1 \geqq -\sqrt{c}, \ y_2 \geqq 0), \text{ or } (y_2 \geqq -\sqrt{c}, \ y_1 \geqq 0), \\ &\qquad \text{or } (y_1 < 0, \ y_2 < 0, \ y_1^2 - y_1 y_2 + y_2^2 \leqq c)\}. \end{aligned}$$

As in the preceding example we see that the likelihood ratio tests of H_0 are admissible. Since this test's maximum Type I error $\bar{\alpha}$ is attained when $\nu_1 = \nu_2 = 0$, by use of tables of the bivariate normal distribution the constant c can be selected to give $\bar{\alpha}$ any desired value.

10. The discrete case. The developments of the preceding sections can also be carried through in the main when $p_\psi(t)$ is a discrete probability distribution function. Regular convergence in the general case is defined (as in [6], p. 134) as follows: $\lim_{i\to\infty} \delta_i = \delta_0$ in the regular sense if, for every bounded subset R of T,

$$\lim_{i\to\infty} \int_R \delta_i(t)\, d\mu(t) = \int_R \delta_0(t)\, d\mu(t).$$

The proof of Theorem 1 above, with $p_\psi(t)$ taken to be discrete, gives

THEOREM 1*. *Every essentially unique (in ξ measure) Bayes' acceptance region is contained in V'.*

We define the class of tests V^* as follows: $\delta \in V^*$ if and only if there exists an open convex set S such that

$$\delta(t) = \begin{cases} 0 & t \text{ an interior point of } S, \\ 1 & t \text{ an exterior point of } S. \end{cases}$$

Then the proof of Theorem 1 above together with Theorem 5.5 of [6] gives

COROLLARY 1*. *If ω' is closed and bounded, V^* is a complete class.*

Since V^* is closed in the sense of regular convergence, from Theorem 5.7 of [6] we obtain

COROLLARY 2*. *V^* is an essentially complete class.*

From the methods of proof of Theorem 2 above we obtain in the discrete case

THEOREM 2*. *V' is contained in $\bar{B}$ if Assumption 1 is satisfied.*

In the discrete case Assumption 2 is always satisfied trivially unless T contains at least one convergent sequence and its limit point. From the methods of proof of Theorems 3 and 4 above, we obtain

THEOREM 3*. *If Assumptions 1 and 2 are satisfied, and if A' and A'' are distinct tests in V_b', then neither test is uniformly as good as the other.*

A modification of Assumption 3 is necessary for the discrete case

ASSUMPTION 3*. For any T' and t' as defined in Assumption 3, ω' contains a sequence of points ψ^i such that

$$\lim_{i\to\infty} p_{\psi^i}(t')/\sum_{t\in T'} p_{\psi^i}(t) = \infty.$$

THEOREM 4*. *If Assumptions 2 and 3* are satisfied, then neither of any pair of distinct tests in V' is uniformly as good as the other.*

Finally, using the methods of proof of Theorems 3 and 4, and defining V_b^* as the subclass of V^* consisting of those tests δ for which the set $\{t \mid \delta(t) < 1\}$ is bounded, we obtain

THEOREM 3a*. *If Assumptions 1 and 2 are satisfied, then neither of any pair of distinct tests in V_b^*, at least one of which is in V_b', is uniformly as good as the other.*

COROLLARY. *If T is bounded and Assumptions 1 and 2 are satisfied, every test in V' is admissible.*

THEOREM 4a*. *If Assumptions 2 and 3* are satisfied, then neither of any pair of distinct tests in V^*, at least one of which is in V', is uniformly as good as the other.*

COROLLARY. *If Assumptions 2 and 3* are satisfied, then every test in V' is admissible.*

EXAMPLE 1. Let $\theta = (p_1, p_2)$, and

$$f_\theta(t) = \frac{n!\, p_1^{t_1} p_2^{t_2}(1 - p_1 - p_2)^{n-t_1-t_2}}{t_1!t_2!(n - t_1 - t_2)!}, \qquad \begin{array}{r} t_1 = 0, 1, \cdots (n - t_2), \\ t_2 = 0, 1, \cdots n. \end{array}$$

We may write

$$f_\theta(t) = p_\psi(t) = \frac{n! \exp \{\psi_1 t_1 + \psi_2 t_2 - n \log (e^{\psi_1} + e^{\psi_2})\}}{t_1!\, t_2!\, (n - t_1 - t_2)!}$$

where $\psi_i = \log [p_i/(1 - p_1 - p_2)]$ for $i = 1, 2$. For

$$H_0: \quad (p_1, p_2) = (p_1^0, p_2^0), \quad p_1^0 > 0, \quad p_2^0 > 0, \quad p_1^0 + p_2^0 < 1,$$

T is bounded, and t_0 is, trivially, continuous on

$$T = \{(t_1, t_2) \mid t_1 = 0, 1, \cdots, (n - t_2); \quad t_2 = 0, 1, \cdots, n\}.$$

Hence Assumptions 1 and 2 are satisfied, and from the Corollary to Theorem 3a* we conclude that an admissible acceptance region for H_0 is given by the common part of T and any convex set.

EXAMPLE 2. Let x_1 and x_2 have independent Poisson distributions, with unknown respective means λ_1 and λ_2. Let $t = (x_1, x_2)$, and consider

$$H_0: \quad \theta = (\lambda_1, \lambda_2) = (\lambda_1^0, \lambda_2^0), \qquad \lambda_i > 0, \quad i = 1, 2.$$

Let $T = (x_1, x_2)$ with $x_i = 0, 1, \cdots$, for $i = 1, 2$. Let A be the common part of T and any convex set. Then as in the preceding example we conclude that A is admissible.

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