DISTRIBUTION OF QUADRATIC FORMS AND SOME APPLICATIONS[1]

By Arthur Grad and Herbert Solomon

Office of Naval Research, Washington, D. C.
and
Teachers College, Columbia University

1. Summary. The authors were prompted by a general problem concerning hit probabilities arising in military operations to seek the distribution of $Q_k = \sum_{i=1}^k a_i x_i^2$, $k = 2, 3$, where the x_i are normally and independently distributed with zero mean and unit variance, $\sum a_i = 1$, and $a_i > 0$. While the distribution of a positive definite quadratic form in independent normal variates has been the subject of several papers in recent years [6], [11], [12], laborious computations are required to prepare from existing results the percentiles of the distribution and a table of hit probabilities. This paper discusses the exact distribution of Q_k and then obtains and tabulates the distributions of Q_2 and Q_3, accurate to four places. Three other approaches to the distributions are discussed and compared with the exact results: a derivation by Hotelling [8], the Cornish-Fisher asymptotic approximation [3], and the approximation obtained by replacing the quadratic form with a chi-square variate whose first two moments are equated to those of the quadratic form—a type of approximation used in components of variance analysis. The exact values and the approximations are given in Tables I and II. The tables have been prepared with the original problem in mind, but also serve as an aid in several problems arising out of quite different contexts, [1], [2], [13]. These are discussed in Section 6.

2. Introduction. A general class of problems arises in military operations when the hit probability of a weapon depends on the combination of two random errors. Suppose random errors in predicted location or predicted position of target and random errors in aim of weapon occur. For purposes of exposition let us limit ourselves to errors in two dimensions. Denote the true position of a target by T, the predicted position, or point of aim, by A, and the point of impact of a weapon aimed at A by I. Let x_1, y_1, be the components of the vector TA and x_2, y_2 the components of the vector AI. If we denote the radius of effectiveness of the weapon by R, then the probability of a hit P is the probability that the resultant vector TI has length no greater than R, or

$$P = P\{x_3^2 + y_3^2 \leqq R^2\}, \tag{1}$$

where $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$.

Received July 2, 1954; revised April 18, 1955.

[1] The tables in this report were computed at Columbia University and Stanford University with the partial support of Office of Naval Research contracts N6onr 271 Task Order II (NR-042-034) and N6onr 251 Task Order III (NR-042-993).

TABLE I

$P(Q_2 \leqq t)$

t	a_2, a_1							
	.5, .5	.6, .4	.7, .3	.8, .2	.9, .1	.95, .05	.99, .01	1, 0
.1	09516	09693	1029	1158	1461	1813	2359	2482
			1028		1345			
			1285		2037			
	1384		1381		2368			
.2	1813	1843	1943	2153	2594	3002	3384	3453
			1942		2465			
			2126		2926			
	2023		2052		3114			
.3	2592	2630	2757	3011	3494	3858	4115	4161
			2756		3399			
			2871		3641			
	2691		2756		3811			
.4	3297	3340	3482	3755	4226	4521	4697	4729
			3481		4180			
			3542		4248			
	3345		3444		4436			
.5	3935	3981	4128	4402	4831	5060	5182	5205
			4127		4835			
			4146		4775			
	3963		4088		4986			
.6	4512	4559	4705	4968	5342	5513	5599	5614
			4705		5387			
			4693		5240			
	4533		4677		5465			
.7	5034	5080	5221	5464	5780	5904	5962	5972
			5221		5854			
			5187		5652			
	5052		5209		5883			
.8	5507	5550	5682	5901	6159	6246	6283	6289
			5683		6251			
			5633		6022			
	5523		5693		6249			
.9	5934	5975	6095	6287	6491	6549	6570	6572
			6096		6592			
			6037		6353			
	5950		6112		6572			

TABLE I—*Continued*

t	a_2, a_1							
	.5, .5	.6, .4	.7, .3	.8, .2	.9, .1	.95, .05	.99, .01	1, 0
1.0	6321	6358	6466	6630	6785	6819	68267	68269
			6467		6886			
			6402		6653			
	6336		6493		6859			
1.5	7769	7785	7826	7866	7858	7830	7801	7793
			7827		7900			
			7770		7781			
	7783		7881		7922			
2.0	8647	8646	8638	8604	8527	8478	8438	8427
			8638		8508			
			8606		8498			
	8749		8788		8700			
3.0	9502	9487	9441	9365	9269	9219	9178	9167
			9441		9234			
			9442		9283			
	9998		10000		9998			
4.0	9817	9802	9761	9698	9624	9585	9553	9545
			9760		9620			
			9770		9643			
	10000		10000		10000			
5.0	9933	9923	9895	9853	9803	9775	9753	9746
			9895		9812			
			9903		9817			
	10000		10000		10000			

First entry in cell is exact to 4 decimal places.
Second entry is Hotelling's result.
Third entry is "components of variance" chi square approximation.
Fourth entry is Cornish-Fisher result.

Now assume that the two random errors are each subject to a bivariate normal distribution with zero means and with covariance matrix $\|{}_p\sigma_{ij}\|$ and $\|{}_a\sigma_{ij}\|$ respectively. Then x_3 and y_3 are components of a vector having a bivariate normal distribution with zero means and covariance matrix $\|{}_p\sigma_{ij} + {}_a\sigma_{ij}\| = \|\lambda_{ij}\|$. For the present, assume the components of each error to be independent; i.e., $\|{}_p\sigma_{ij}\|$ and $\|{}_a\sigma_{ij}\|$ are diagonal. This restriction, which is not essential, implies that x_3 and y_3 are independently distributed. If $x = \lambda_{11}^{-1/2} x_3$ and $y = \lambda_{22}^{-1/2} y_3$, then x^2 and y^2 each have a chi-square distribution with one degree of freedom. We may then write

$$P = P\{a_1x^2 + a_2y^2 \leqq t\} \tag{2}$$

TABLE II

$P\{Q_3 \leqq t\}$

t	a_3, a_2, a_1								
	$\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$,	.4,. 3, .3	.4, .4, .2	.5, .3, .2	.6, .2, .2	5, .4, .1	.6, .3, .1	.7, .2, .1	.8, .1, .1
.1	03997	04146	04313	04385	05035	05169	05421	06062	07419
		04048		04377				05564	05773
		0470		0602				1150	1548
		0697		0721				0945	1544
.2	10357	1053	1094	1123	1217	1282	1338	1477	1803
		1047		1122				1402	1483
		1083		1275				1971	2416
		1220		1265				1633	2336
.3	17457	1763	1830	1873	2026	2081	2162	2357	2758
		1763		1872				2296	2458
		1768		1985				2716	3155
		1849		1916				2411	3137
.4	24700	2491	2571	2624	2803	2852	2951	3179	3625
		2491		2623				3159	3406
		2474		2692				3397	3805
		2529		2617				3200	3886
.5	31773	3201	3287	3346	3541	3570	3679	3923	4353
		3201		3346				3952	4273
		3172		3375				4016	4381
		3216		3319				3946	4596
.6	38507	3875	3961	4023	4223	4228	4340	4584	4979
		3875		4024				4663	5037
		3841		4020				4580	4897
		3880		3992				4623	5141
.7		4505	4587	4649	4843	4825	4936	5169	5515
		4505		4650					
		4471		4621				4909	5360
		4506		4620				5214	5649
.8	50637	5086	5161	5220	5402	5363	5469	5683	5974
		5086		5222				5829	6239
		5056		5175				5555	5776
		5085		5195				5751	6088
.9		5618	5683	5739	5902	5848	5945	6136	6371
		5618		5740					
		5594		5682				5975	6152
		5615		5718				6175	6471

TABLE II—*Continued*

t	a_3, a_2, a_1								
	⅓, ⅓, ⅓,	.4, .3, .3	.4, .4, .2	.5, .3, .2	.6, .2, .2	.5, .4, .1	.6, .3, .1	.7, .2, .1	.8, .1, .1
1.0	60837	6102	6156	6206	6349	6282	6370	6535	6717
		6102		6207				6697	7056
		6083		6143				6355	6491
		6097		6189				6619	6806
1.5		7881	7884	7901	7935	7863	7895	7935	7930
		7881		7901					
		7885		7848				7776	7766
		7876		7894				8042	8008
2.0	88839	8879	8853	8844	8808	8770	8760	8723	8663
		8879		8844				8659	8527
		8889		8820				8636	8558
		8972		8931				8992	8888
3.0	97071	9698	9668	9645	9577	9591	9552	9477	9378
		9698		9645				9394	9270
		9702		9650				9477	9379
		10000		10000				9933	10000
4.0	99262	9920	9905	9888	9841	9863	9831	9775	9703
		9920		9888				9763	9734
		9921		9896				9794	9724
		10000		10000				10000	10000
5.0	99818	9979	9973	9963	9938	9954	9935	9900	9855
		9979		9964				9916	9897
		9979		9969				9917	9874
		10000		10000				10000	10000

First entry in cell is exact to 4 decimal places.
Second entry is Hotelling's result.
Third entry is "components of variance" chi square approximation.
Fourth entry is Cornish-Fisher result.

where $\sigma^2 = \lambda_{11} + \lambda_{22}$, $a_i = \lambda_{ii}/\sigma^2$ and $t = R^2/\sigma^2$. In the three-dimensional situation, we get by the same argument

$$P = P\{a_1x^2 + a_2y^2 + a_3z^2 \leqq t\}, \tag{3}$$

where this time $\sigma^2 = \lambda_{11} + \lambda_{22} + \lambda_{33}$. Similarly, if we leave physical reality, we obtain in k dimensions

$$P = P\left\{\sum_{i=1}^{k} a_i x_i^2 \leqq t\right\} = P\{Q_k \leqq t\} \tag{4}$$

where $\sigma^2 = \sum_1^k \lambda_{ii}$. Now remove the restriction of independence of errors; that is, let the covariance matrix be an arbitrary positive definite matrix. Then there exists a real non-singular linear transformation [4], $Y = CX$, such that the covariance matrix in the new variables y_i is the unit matrix, and Q_k has the form $\sum_1^k \alpha_i y_i^2$, where the α_i are the roots of the determinantal equation $|A - \alpha\Lambda^{-1}| = 0$, and are all positive, A is the matrix of the coefficients of Q_k considered as a form in the variables x_i, and Λ is the covariance matrix $\{\lambda_{ij}\}$ in these variables. Thus in this paper only (4) is discussed since all other situations can be reduced to it.

3. Exact distribution. Consider the positive definite quadratic form $Q_k = \sum_1^k a_i x_i^2$, where the x_i are normally and independently distributed about zero with unit variance, $\sum a_i = 1$, and $0 < a_i \leqq a_{i+1}$. Denote by $F_k(t)$ the distribution function $F_k(t) = P\{Q_k \leqq t\}$, and by $f_k(t)$ the probability density. Then the Laplace transform $\phi_k(p)$ of $f_k(t)$ is

$$\phi_k(p) = \prod_{j=1}^{k} (1 + 2a_j p)^{-1/2} \tag{5}$$

From this, $f_k(t)$ and $F_k(t)$ can be obtained in various forms. The authors are including only those which appear most efficient for computing purposes. The following approach was found most useful. Inverting the transform (5) we obtain

$$f_k(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{tp} \phi_k(p)\, dp. \tag{6}$$

We now apply Cauchy's theorem to the integrand in (6) taken along the closed contour from $-iR$ to iR along the imaginary axis, from iR to $-R$ along a quarter circle around the origin, from $-R$ to -1 and back along the negative real axis with small clockwise semicircular identations of radius r to avoid the singularities $-\frac{1}{2}a_j$, and from $-R$ back to $-iR$ along a quarter circle around the origin. Letting $R \to \infty$ and $r \to 0$, we obtain

$$f_{2k}(t) = \frac{1}{\pi} \sum_{n=1}^{k} (-1)^{k-n} \int_{-1/2a_{2n-1}}^{-1/2a_{2n}} e^{tp} \phi_{2k}(p)\, dp, \tag{7}$$

$$\begin{aligned} f_{2k+1}(t) = {} & \frac{(-1)^k}{\pi} \int_{-\infty}^{-1/2a_1} e^{tp} \phi_{2k+1}(p)\, dp \\ & + \frac{1}{\pi} \sum_{n=1}^{k} (-1)^{k-n} \int_{-1/2a_{2n}}^{-1/2a_{2n+1}} e^{tp} \phi_{2k+1}(p)\, dp. \end{aligned} \tag{8}$$

We now let $c_j = 1/a_j$, and make the changes of variables

$$p = p_1(x, t) = -\tfrac{1}{2}c_1 - x^2/t, \qquad (-\infty < p < -\tfrac{1}{2}c_1),$$

$$p = p_n(x) = \tfrac{1}{4}(c_{n-1} - c_n)x - \tfrac{1}{4}(c_{n-1} + c_n), \qquad (-\tfrac{1}{2}c_{n-1} < p < \tfrac{1}{2}c_n).$$

For even index we obtain

$$f_{2k}(t) = \frac{(-1)^k}{2\pi} \left\{\prod_{j=1}^{2k} \sqrt{c_j}\right\} \int_{-1}^{1} \sum_{n=1}^{k} (-1)^n G_{2n}(x, t, 2k) \frac{dx}{\sqrt{1 - x^2}}, \tag{9}$$

where

$$G_n(x, t, k) = e^{t p_n(x)} \prod_{m=1, m \neq n-1, n}^{k} [c_m + 2p_n(x)]^{-1/2} \tag{10}$$

Integrating (9), we get

$$F_{2k}(t) = 1 + \frac{(-1)^k}{2\pi}\left\{\prod_{j=1}^{2k} \sqrt{c_j}\right\} \int_{-1}^{1} \sum_{n=1}^{k} (-1)^n \frac{G_{2n}(x, t, 2k)}{P_{2n}(x)} \frac{dx}{\sqrt{1 - x^2}} \tag{11}$$

Similarly, for odd index,

$$\begin{aligned} f_{2k+1}(t) &= \frac{(-1)^k}{2\pi}\left\{\prod_{j=1}^{2k+1} \sqrt{c_j}\right\} \\ &\qquad \int_{-1}^{1} \sum_{n=1}^{k} (-1)^n G_{2n+1}(x, t, 2k+1) \frac{dx}{\sqrt{1 - x^2}} + r_{2k+1}(t), \end{aligned} \tag{12}$$

where

$$r_{2k+1}(t) = \frac{(-1)^k}{2\pi}\left\{\prod_{j=1}^{2k+1} \sqrt{c_j}\right\} \left(\frac{t}{2}\right)^{k-\frac{1}{2}} e^{-\frac{1}{2}c_1 t} \int_{-\infty}^{\infty} H(x, t, k) e^{-x^2}\, dx, \tag{13}$$

and

$$H(x, t, k) = \prod_{m=1}^{2k} [x^2 + \tfrac{1}{2}(c_1 - c_{m+1})t]^{-\frac{1}{2}} \tag{14}$$

Integrating (8), we get

$$\begin{aligned} F_{2k+1}(t) &= 1 + \frac{(-1)^k}{2\pi}\left\{\prod_{j=1}^{2k+1} \sqrt{c_j}\right\} \\ &\qquad \int_{-1}^{1} \sum_{n=1}^{k} (-1)^n \frac{G_{2n+1}(x, t, 2k+1)}{p_{2n+1}(x)} \frac{dx}{\sqrt{1 - x^2}} + R_{2k+1}(t), \end{aligned} \tag{15}$$

where

$$R_{2k+1}(t) = \frac{(-1)^k}{2\pi}\left\{\prod_{j=1}^{2k+1} \sqrt{c_j}\right\} \left(\frac{t}{2}\right)^{k-\frac{1}{2}} e^{-\frac{1}{2}c_1 t} \int_{-\infty}^{\infty} \frac{H(x, t, k)}{p_1(x, t)} e^{-x^2}\, dx. \tag{16}$$

The integrals over the interval $(-1, 1)$ are readily computed using the quadrature formula [16]

$$\int_{-1}^{1} f(x) \frac{dx}{\sqrt{1 - x^2}} = \lim_{n \to \infty} \frac{\pi}{n} \sum_{i=1}^{n} f(x_i^{(n)}), \tag{17}$$

where $x_i^{(n)}$ are the zeros of the Tchebycheff polynomials $T_n(x)$ of degree n. Similarly, the zeros $y_i^{(n)}$ and Christoffel numbers $\alpha_i^{(n)}$ of the Hermite polynomials [14] can be used in computing $r_k(t)$ and $R_k(t)$ with the quadrature formula [14], [16]

$$\int_{-\infty}^{\infty} e^{-y^2} f(y)\, dy = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i^{(n)} f(y_i^{(n)}). \tag{18}$$

These are usually small unless t is also small, or the two largest coefficients c_1 and c_2 are almost equal. Except under these conditions, they can generally be shown to be negligible by the inequalities

$$|r_{2k+1}(t)|^2 < \frac{1}{2\pi} c_1 \left\{ \prod_{m=1}^{2k} \frac{c_{m+1}}{c_1 - c_{m+1}} \right\} t^{-1} e^{-c_1 t}, \tag{19}$$

$$|R_{2k+1}(t)|^2 < \frac{2}{\pi c_1} \left\{ \prod_{m=1}^{2k} \frac{c_{m+1}}{c_1 - c_{m+1}} \right\} t^{-1} e^{-c_1 t}, \tag{20}$$

which are obtained from (13) and (16) by making use of

$$|H(x, t, k)| \leqq \prod_{m=1}^{2k} \{\tfrac{1}{2}(c_1 - c_{m+1})t\}^{-\frac{1}{2}}, \qquad |p_1(x, t,)| \geqq \tfrac{1}{2}c_1 .$$

For the original two-dimensional problem, we obtain from (9) and (11),

$$f_2(t) = \frac{1}{2\pi} \sqrt{c_1 + c_2}\, e^{-\frac{1}{2}(c_1+c_2)t} \int_{-1}^{1} e^{\frac{1}{2}(c_1-c_2)tx} \frac{dx}{\sqrt{1 - x^2}}, \tag{21}$$

$$F_2(t) = 1 - \frac{2}{\pi} \sqrt{c_1 + c_2}\, e^{-\frac{1}{2}(c_1+c_2)t} \int_{-1}^{1} \frac{e^{\frac{1}{2}(c_1-c_2)tx}}{(c_1 + c_2) - (c_1 - c_2)x} \frac{dx}{\sqrt{1 - x^2}}, \tag{22}$$

which can be simplified to

$$f_2(t) = \tfrac{1}{2}\sqrt{c_1 + c_2}\, e^{-\frac{1}{2}(c_1+c_2)t} I_0[\tfrac{1}{4}(c_1 - c_2)t], \tag{23}$$

$$F_2(t) = \frac{2}{\sqrt{c_1 + c_2}} \int_0^{\frac{1}{4}(c_1+c_2)t} e^{-x} I_0[\sqrt{1/c_2 - 1/c_1}\, x]\, dx, \tag{24}$$

where I_0 is the modified Bessel function of order zero. Although (23) is analytically preferable to (21), (22) is easier to evaluate numerically than (24) except for very small values of t.

The case $k = 3$ applies to the original problem in three dimensions. This time (12) and (15) become

$$f_3(t) = \frac{1}{\pi} \sqrt{\frac{c_1 c_2 c_3}{2}}\, e^{-\frac{1}{2}(c_2+c_1)t} \int_{-1}^{1} \frac{e^{-\frac{1}{2}(c_2-c_1)tx}}{\sqrt{2c_3 - (c_2 + c_1) - (c_2 - c_1)x}} \frac{dx}{\sqrt{1 - x^2}} + r_3(t), \tag{25}$$

and

$$F_3(t) = 1 - \frac{1}{\pi} \sqrt{8c_1 c_2 c_3}\, e^{-\frac{1}{2}(c_2+c_1)t} \int_{-1}^{1} \frac{e^{-\frac{1}{2}(c_2-c_1)tx}}{[(c_2 + c_1) + (c_2 - c_1)x]\sqrt{2c_3 - (c_2 + c_1) - (c_2 - c_1)x}} \frac{dx}{\sqrt{1 - x^2}} + R_3(t) \tag{26}$$

where

$$r_3(t) = -\frac{1}{\pi}\sqrt{\tfrac{1}{8}c_1c_2c_3}\,t^{\frac{1}{2}}e^{-c_3t/2}\int_{-\infty}^{\infty}\frac{e^{-x^2}\,dx}{\sqrt{[x^2+\frac{1}{2}(c_3-c_1)t][x^2+\frac{1}{2}(c_3-c_2)t]}}, \tag{27}$$

and

$$R_3(t) = \frac{1}{\pi}\sqrt{\tfrac{1}{8}c_1c_2c_3}\,t^{3/2}e^{-c_3t/2}\int_{-\infty}^{\infty}\frac{e^{-x^2}\,dx}{(x^2+c_3t/2)\sqrt{[x^2+\frac{1}{2}(c_3-c_1)t][x^2+\frac{1}{2}(c_3-c_2)t]}}. \tag{28}$$

Numerical evaluation of $f_k(t)$ and $F_k(t)$ becomes more difficult if the constants c_i are almost equal. In that case, however, an as yet unpublished method of Hotelling [8] becomes effective. This will be discussed in the next section. On the other hand, for $f_3(t)$, if two of the constants, say c_j, actually coincide, then the problem simplifies and we obtain as the inverse transform of (5), [5]

$$f_3(t) = \tfrac{1}{2}c_j\sqrt{\frac{c_i}{c_i-c_j}}\,e^{-c_jt/2}\operatorname{erf}\sqrt{\tfrac{1}{2}(c_i-c_j)t}. \tag{29}$$

Hence

$$F_3(t) = I\left(\frac{1}{\sqrt{2}}c_i\,t,\,-\tfrac{1}{2}\right)-\sqrt{\frac{c_i}{c_i-c_j}}\,e^{-c_jt/2}\operatorname{erf}\sqrt{\tfrac{1}{2}(c_i-c_j)t}, \tag{30}$$

where $I(u, p)$ is the incomplete gamma function as tabulated in [10]. The first entry of each cell in Tables I and II was obtained from the quadrature formulas given above and is correct to four decimal places.

There is an interesting relationship between the distribution of Q_2 and the distribution of the measure of the random set given in [15]. If ${}_a\sigma_{ij} = \sigma_a^2$ for $i = j$ and ${}_a\sigma_{ij} = 0$ for $i \neq j$ and the vector TA mentioned early in the paper is constant, say D, the graph labelled Figure 1 in [15] gives the desired probability if we consider the abscissa values equal to D/σ_a and the ordinate values equal to R/σ_a. Let us now return to our present problem but add the further restriction ${}_p\sigma_{ij} = \sigma_p^2$ for $i = j$ and ${}_p\sigma_{ij} = 0$ for $i \neq j$. Then the probability density of D/σ_p, $h(D/\sigma_p)$, is

$$h\left(\frac{D}{\sigma_p}\right) = \frac{D}{\sigma_p}e^{-\frac{1}{2}(D/\sigma_p)^2}\,d\left(\frac{D}{\sigma_p}\right) \tag{31}$$

and

$$P = P\left\{Q_2 \leq \frac{R^2}{2(\sigma_a^2+\sigma_p^2)}\right\} = \int_0^{\infty} g\left(\frac{R}{\sigma_a}\,\middle|\,\frac{D}{\sigma_a}\right)h\left(\frac{D}{\sigma_p}\right) \tag{32}$$

where $g(R/\sigma_a \mid D/\sigma_a)$ is the probability read from the graph in [15] and the coefficients of Q_2 are now both equal to $\frac{1}{2}$. As an illustration, consider the following four situations: (a) $R/\sigma_a = 2$, $\sigma_p^2/\sigma_a^2 = 3$; (b) $R/\sigma_a = 2$, $\sigma_p^2/\sigma_a^2 = 1$; (c) $R/\sigma_a = 3$, $\sigma_p^2/\sigma_a^2 = 2$; (d) $R/\sigma_a = 3$, $\sigma_p^2/\sigma_a^2 = 1$; then in the table immediately following we get the top entries from Table I, and the bottom entries by numerical integration of (28).

(a)	(b)	(c)	(d)
.3935	.6321	.7769	.8883
.3971	.6328	.7767	.8955

Thus, since only two place accuracy at best could be obtained by reading $g(R/\sigma_a \mid D/\sigma_a)$ from the graph, a rather simple numerical integration yields values extremely close to the exact values.

4. Hotelling's method.[2] Let $2q = Q_k$ and modify Q_k by requiring $\sum a_i = k = 2m$ so that in our cases of special interest $m = 1$ or $\frac{3}{2}$. The a_i are now the ratios of the latent roots of Q_k to k times the trace of the matrix of Q_k where k is rank. Then Hotelling states that the density of q is,

$$f(q) = \frac{q^{m-1}e^{-q}}{\Gamma(m)} \sum_{r=0}^{\infty} b_r L_r(q), \tag{33}$$

where

$$b_r = \frac{r!\,\Gamma(m)}{\Gamma(m+r)} \int_0^{\infty} f(q) L_r(q)\, dq, \tag{34}$$

and $L_r(q)$ is a Laguerre polynomial defined by

$$L_r(q) = \sum_{t=0}^{r} \binom{r+m-1}{r-t} \frac{(-q)^t}{t!}. \tag{35}$$

Now define

$$u_r = \sum_{j=1}^{k} (a_j - 1)^r. \tag{36}$$

[2] In a letter to one of the authors [8] in November, 1950, Hotelling outlined his method for obtaining the distribution of quadratic forms. This letter was in response to a query regarding a talk Hotelling gave in a seminar attended by one of the authors in Berkeley in 1947. Mention of this research also appears in an abstract by Hotelling in *Ann. Math. Stat.*, Vol. 19 (1948), p. 119.

Then

$$f(q) = \frac{1}{\Gamma(m)} q^{m-1} e^{-q} \left\{ 1 + \frac{u_2}{4}\left[1 - \frac{2q}{m} + \frac{q^2}{m(m+1)}\right] \right.$$

$$- \frac{u_3}{3!}\left[1 - \frac{3q}{m} + \frac{3q^2}{m(m+1)} - \frac{q^3}{m(m+1)(m+2)}\right]$$

$$+ \frac{3\left(u_4 + \frac{u_2^2}{4}\right)}{4!}\left[1 - \frac{4q}{m} + \frac{6q^2}{m(m+1)} - \frac{4q^3}{m(m+1)(m+2)} \right. \tag{37}$$

$$\left. + \frac{q^4}{m(m+1)(m+2)(m+3)}\right]$$

$$- \frac{12u_5 + 5u_2u_3}{5!}\left[1 - \frac{5q}{m} + \frac{10q^2}{m(m+1)} - \frac{10q^3}{m(m+1)(m+2)} \right.$$

$$\left.\left. + \frac{5q^4}{m(m+1)\cdots(m+3)} - \frac{5q^5}{m(m+1)\cdots(m+4)}\right]\right\}$$

+ further terms requiring higher moments of the normal distribution.

Rearranging Hotelling's terms to make optimum use of the Hartley-Pearson Tables [7], we get

$$\begin{aligned} F(t) = {} & P\{x_2^2 \leqq 2t\}\cdot[1 + d_2 - d_3 + d_4 - d_5] \\ & + P\{x_4^2 \leqq 2t\}\cdot[-2d_2 + 3d_3 - 4d_4 + 5d_5] \\ & + P\{x_6^2 \leqq 2t\}\cdot[d_2 - 3d_3 + 6d_4 - 10d_5] \\ & + P\{x_8^2 \leqq 2t\}\cdot[d_3 - 4d_4 + 10d_5] \\ & + P\{x_{10}^2 \leqq 2t\}\cdot[d_4 - 5d_5] \\ & + P\{x_{12}^2 \leqq 2t\}\cdot[d_5] \end{aligned} \tag{38}$$

where

$$d_2 = \frac{u_2}{4}, \qquad d_3 = \frac{u_3}{6}, \qquad d_4 = \tfrac{1}{8}(u_4 + \tfrac{1}{4}u_2^2), \qquad d_5 = \tfrac{1}{120}(12u_5 + 5u_2u_3),$$

and x_n^2 is a chi-square variate with n degrees of freedom. The values obtained by this method using (34) are quite accurate. Using the fixed number of terms in (34), the departure from the exact value depends on the variance of the a_i's. This is noted by a glance at the second entry in each cell of Tables I and II having more than one entry. Thus this method complements the method given in Section 3 precisely in those cases where the most numerical difficulty is experienced; namely, when the variance in the a_i's is small.

5. Approximations. Where a third entry appears in a cell of Tables I and II, it is an approximation obtained in the following way. Let $Q_k = cx_n^2$; this is an approximating device often used in components of variance analysis. Then, equating the first two moments, we get

$$cn = \sum_{i=1}^{k} a_i = 1, \qquad c^2 n = \sum_{i=1}^{k} a_i^2.$$

Thus Q_k is approximated by $(\sum_1^k a_i^2)x^2$ where x^2 has $n = 1/\sum_1^k a_i^2$ degrees of freedom. To avoid the interpolation caused by fractional degrees of freedom we can employ the Wilson-Hilferty approximation [17] which states that given a chi-square variate with n degrees of freedom, say χ_n^2, then $(\chi^2/n)^{1/3}$ is approximately normally distributed with mean $(1 - 2/9n)$ and variance $2/9n$; thus we may write

$$P\{Q_k \leqq t\} = P\left\{\left(1 - \frac{2}{9n} + x\sqrt{2/9n}\right)^3 \leqq t\right\} \tag{39}$$

as a modified approximation where x is normally distributed with zero mean and unit variance. Finally we get

$$P\{Q_k \leqq t\} = P\left\{x \leqq \frac{t^{1/3} - (1 - \frac{2}{9}\sum_1^k a_i^2)}{\sqrt{\frac{2}{9}\sum_1^k a_i^2}}\right\}. \tag{40}$$

This result, together with Kelley's Tables [9], was used to obtain the third entry in the cells of the tables wherever they appear.

Where a fourth entry appears in a cell of the tables, it is an approximation obtained from the Cornish-Fisher [3] asymptotic expansion of Q_k in terms of normal variable. This approximation requires the cumulants of Q_k, but these are easy to obtain from the cumulants of the chi-square variate with one degree of freedom by applying the additive properties of cumulants. Computation of the values in Tables I and II is based on all terms in the asymptotic expansion of orders through $1/k^2$.

6. Applications. In discussing applications there is, of course, the obvious one which motivated this paper. As an illustration, assume ${}_a\sigma_{11} = 100$, ${}_a\sigma_{22} = 400$, ${}_p\sigma_{11} = 100$, ${}_p\sigma_{22} = 1400$, and $R = 40$. In this case the usual assumption of circular symmetry is certainly not realistic. Here $a_1 = .1$, $a_2 = .9$, and $t = .8$. Thus the probability of a hit is read as .6159 from Column 5 in Table I. Moreover, Tables I and II make it possible to compare the relative effects of changes in weapon radius with changes in aiming and location errors.

In [2] it is demonstrated that the usual chi-square tests for goodness of fit do not have a limiting chi-square distribution when the maximum likelihood estimates of the parameters are based on the original observations rather than on the cell frequencies. The asymptotic distribution in this situation is that of

$$\sum_{i=1}^{j-s-1} y_i^2 + \sum_{i=j-s}^{j-1} \theta_i y_i^2, \tag{41}$$

where j is the number of cells, s is the number of parameters to be estimated, and the coefficients θ_i are between zero and one and are the roots of a determinantal equation. In the usual "goodness of fit" situation in statistics, distributions rarely contain more than two parameters to be estimated from the data. Thus Tables I and II are singularly appropriate if the number of cells is kept down. In an illustration given in [2],

$$P = P\{x_1^2 + .8x_2^2 + .2x_3^2 \geqq 3.84\} \tag{42}$$

is desired, and $P = .12$ is given as a lower bound. This can be quickly modified so that Table II can be used, for dividing through by two in (38) we get

$$P = P\{.5x_1^2 + .4x_2^2 + .1x_3^2 \geqq 1.92\}. \tag{43}$$

From an Aitken seven point interpolation in the (.5, .4, .1) column in Table II, we get $P = .1344$.

In [1], the limiting distribution of $n\omega^2$ is obtained as the distribution of the quadratic form $Q_\infty = \sum_1^\infty a_i x_i^2$ where $a_i = 1/i^2\pi^2$, and ω^2 is the von Mises criterion for goodness of fit between a sample cumulative distribution function and a specified population distribution function. In [13], it is shown that a simple variant of the ω^2 criterion for the two-sample test has the same limiting distribution. While a table of this distribution is given in [1] it should be possible to use Table II to some advantage, even though this means neglecting all terms from $i = 4$ onwards. Since $\sum_1^\infty a_i = \frac{1}{6} = .1667$ and $\sum_1^3 a_i = 49 / 36\pi^2 = .1379$, a reasonable upper bound should be given by Table II. For example take $t = .046$, $t = .101$, and $t = .405$, then the table in [1] yields .10, .42, and .93 respectively while from Table II we get using

$$P\left\{\frac{x_1^2}{\pi^2} + \frac{x_2^2}{4\pi^2} + \frac{x_3^2}{9\pi^2} \leqq t\right\} = P\left\{\frac{36}{49}x_1^2 + \frac{9}{49}x_2^2 + \frac{4}{49}x_3^2 \leqq \frac{36\pi^2}{49}t\right\} \tag{44}$$

that the probabilities are .28, .54, and .94 respectively. These values are obtained by interpolation and are correct to two places. However, the upper bound is not too sharp when P is small. Also Table II is constructed with t as the argument while the table in [1] has P as the argument and thus may be more useful in some contexts and, of course, less in others.

REFERENCES

[1] T. W. Anderson and D. A. Darling, "Asymptotic theory of certain 'goodness of fit' criteria based on stochastic processes," *Ann. Math. Stat.*, Vol. 23 (1952), pp. 193–211.

[2] H. Chernoff and E. Lehmann, "The use of maximum likelihood estimates in chi square tests for goodness of fit," *Ann. Math. Stat.*, Vol. 25 (1954), pp. 573–578.

[3] E. A. Cornish and R. A. Fisher, "Moments and cumulants in the specification of distributions," *Revue de l'Institut International de Stat.*, Vol. 4 (1937), pp. 307–320.

[4] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. I, Interscience, New York, 1953.

[5] *Tables of Integral Transforms*, Vol. I, edited by A. Erdélyi, McGraw-Hill, New York, 1954.

[6] J. Gurland, "Distribution of quadratic forms and ratios of quadratic forms," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 416–427.

[7] H. O. Hartley and E. S. Pearson, "Tables of the χ^2 integral and of the cumulative Poisson distribution," *Biometrika*, Vol. 37 (1950), pp. 313–325.

[8] H. Hotelling, Private communication to authors, November, 1950.

[9] T. L. Kelley, *The Kelley Statistical Tables*, Harvard University Press, 1948.

[10] *Tables of the Incomplete Gamma Function*, edited by Karl Pearson, Biometrika Office, London, 1946.

[11] H. Robbins, "The distribution of a definite quadratic form," *Ann. Math. Stat.*, Vol. 19 (1948), pp. 266–270.

[12] H. Robbins and E. J. G. Pitman, "Applications of the method of mixtures to quadratic forms in normal variates," *Ann. Math. Stat.*, Vol. 20 (1949), pp. 552–560.

[13] M. Rosenblatt, "Limit theorems associated with variants of the von Mises statistic," *Ann. Math. Stat.*, Vol. 23 (1952), pp. 617–623.

[14] H. E. Salzer, R. Zucker, and R. Capuano, "Table of the zeros and weight factors of the first twenty hermite polynomials," *Journal of Research of the National Bureau of Standards*, Vol. 48, No. 2, February 1952, pp. 111–116.

[15] H. Solomon, "Distribution of the measure of a random two-dimensional set," *Ann. Math. Stat.*, Vol. 24 (1953), pp. 650–656.

[16] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society, Colloquium Publications, Vol. 23, New York, 1939.

[17] E. B. Wilson and M. M. Hilferty, "The distribution of chi-square" *National Academy of Sciences*, Vol. 17 (1931), pp. 694–698.