# NOTES

## FURTHER REMARK ON THE MAXIMUM NUMBER OF CONSTRAINTS OF AN ORTHOGONAL ARRAY

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**1. Summary.** R. C. Bose and K. A. Bush [1] showed how to make use of the maximum number of points, no three collinear, in finite projective spaces for the construction of orthogonal arrays. In particular, this enabled them to construct an orthorgonal array (81, 10, 3, 3). They proved, on the other hand, that in the case considered the maximum number of constraints does not exceed 12. Hence they state, "We do not know whether we can get 11 or 12 constraints in any other way." A partial solution to this problem was given by the author [2]. It was shown that the number of constraints cannot exceed 11. The purpose of this paper is to give a complete solution to the above stated problem, namely, to prove that no way exists which could give a number of constraints, of the considered orthogonal array, greater than ten. As a consequence of the proof it follows also that any orthogonal array with ten constraints satisfies a unique algebraic solution. It is not known, however, whether the arrays constructed by the geometrical method form the totality of orthogonal arrays of the considered type.

**2. Introduction.** The proof is based on an algebraic property of orthogonal arrays, pointed out by Bose and Bush [1]. Let $n_{ij}^k$ denote the number of columns belonging to an array consisting of $k$ rows that have $j$ coincidences ($j$ elements equal) with the $i$th column. A necessary condition for an array $(\lambda s^t, k, s, t)$ to be orthogonal is that, whatever be the number $h$ such that $0 \leqq h \leqq t$, the following equalities hold.

$$\sum_{j=0}^{k} n_{ij}^k c_h^j = c_h^k(\lambda s^{t-h} - 1), \qquad i = 1, 2, \cdots, s^t,$$

where the $c$'s are binomial coefficients.

In the case considered the equalities become, for $i = 1, 2, \cdots, 81$,

$$(1) \qquad \begin{aligned} &\sum_{j=0}^{k} n_{ij}^k = 80, && \sum_{j=0}^{k} j(j-1)n_{ij}^k = 8k(k-1), \\ &\sum_{j=0}^{k} jn_{ij}^k = 26k, && \sum_{j=0}^{k} j(j-1)(j-2)n_{ij}^k = 2k(k-1)(k-2). \end{aligned}$$

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Use will be made of Lemma 1 proved in [2] which asserts that an orthogonal array (81, 10, 3, 3) for which $n_{1j}^{10} = 0$ for $j \geqq 5$ cannot be extended to an eleven-rowed orthogonal array. It will be assumed without loss of generality that the first column consists of zeros only.

## 3. Derivations.

LEMMA 1. *If $k = 4$, then $n_{i4}^{4}$ is equal to zero or two for $i = 1, 2, \cdots, 81$.*

PROOF. Equations (1) become in this case

$$\begin{aligned} n_{i0}^{4} + n_{i1}^{4} + n_{i2}^{4} + n_{i3}^{4} + n_{i4}^{4} &= 80, \\ n_{i1}^{4} + 2n_{i2}^{4} + 3n_{i3}^{4} + 4n_{i4}^{4} &= 104, \\ 2n_{i2}^{4} + 6n_{i3}^{4} + 12n_{i4}^{4} &= 96, \\ 6n_{i3}^{4} + 24n_{i4}^{4} &= 48, \qquad i = 1, 2, \cdots, 81. \end{aligned}$$

Thus an orthogonal array with four constraints has to satisfy one of the following solutions.

| Solution | $n_{ij}^{4}$ | | | | |
|---|---|---|---|---|---|
| | $n_{i4}^{4}$ | $n_{i3}^{4}$ | $n_{i2}^{4}$ | $n_{i1}^{4}$ | $n_{i0}^{4}$ |
| I | 0 | 8 | 24 | 32 | 16 |
| II | 1 | 4 | 30 | 28 | 17 |
| III | 2 | 0 | 36 | 24 | 18 |

It is seen that Lemma 1 reduces to showing that solution II is impossible. Clearly it will be enough to show that Lemma I holds for $i = 1$. Let us assume furthermore that the first three rows have the form

000000000000000000000000000
000000000111222111222111222
000111222000000111111222222

111111111111111111111111111
000000000111222111222111222
000111222000000111111222222

222222222222222222222222222
000000000111222111222111222
000111222000000111111222222

where the middle and last thirds of each row are printed below the first third. Assume now for the sake of the proof that the fourth row has a zero in the first, second, and fourth columns. Then the remaining zeros will be distributed as follows.

| Serial number of the columns | Number of zeros in fourth row |
|---|---|
| 10–15 | 1 |
| 16–21 | 2 |
| 22–27 | 3 |
| 28–30, 55–57 | 1 |
| 31–33, 58–60 | 2 |
| 34–36, 61–63 | 3 |
| 37–42, 64–69 | 5 |
| 43–54, 70–81 | 7 |
| Total | 24 |

Consider now the fourth column of the array. The assumptions made imply that $n_{44}^4 = 0$. Hence for $i = 4$ solution I will have to hold. This means that $n_{40}^4 = 16$ and $n_{42}^4 = 24$. It will be shown that $n_{40}^4 = 16$ implies $n_{42}^4 = 26$, which is impossible. Let us find first the position of the sixteen columns which have no coincidences with the fourth column. It follows from the distribution of zeros that seven such columns will be found among the columns 37–42 and 64–69. Thus the remaining nine columns will have to be among the columns 49–54, 70–81. This in turn implies that the fourth row will have to have three zeros among the columns 49–54 and 76–81.

Let us count now the number of columns which have two coincidences with the fourth column.

| Serial number of the columns | Number of columns having two coincidences with the fourth column |
|---|---|
| 3 | 1 |
| 5–9 | 5 |
| 10–15 | 1 |
| 16–21 | 4 |
| 22–27 | 3 |
| 28–30, 55–57 | 1 |
| 31–33, 58–60 | 4 |
| 34–36, 61–63 | 3 |
| 43–48, 70–75 | 4 |
| Total | 26 |

This concludes the proof of the lemma.

THEOREM 1. *The number of coincidences of any two columns of a five-rowed orthogonal array is less than five provided that* $\lambda = s = t = 3$.

PROOF. Suppose that there exists a column which has five coincidences with some of the remaining columns of the array. We may assume that this column is the first column of the orthogonal array. Consider now the solutions of equations (1) for $i = 1$ and $k = 5$. It is easy to see that there are only four sets of such solutions. Namely,

| Solution | $n^5_{ij}$ | | | | | |
|---|---|---|---|---|---|---|
| | $n^5_{15}$ | $n^5_{14}$ | $n^5_{13}$ | $n^5_{12}$ | $n^5_{11}$ | $n^5_{10}$ |
| I′ | 1 | 2 | 2 | 52 | 7 | 16 |
| II′ | 2 | 0 | 0 | 60 | 0 | 18 |
| III′ | 1 | 1 | 6 | 46 | 11 | 15 |
| IV′ | 1 | 0 | 10 | 40 | 15 | 14 |

It will be shown that none of these solutions can give rise to an orthogonal array.

Let us assume without loss of generality that the first three rows are the same as considered in Lemma 1 and that the fourth and fifth rows have also a zero in the first column. Consider first solution I′. In this case $n^5_{15} = 1$ and $n^5_{14} = 2$. Thus the five-rowed array contains a three-rowed subarray in which $n^3_{13} \geqq 3$; but this contradicts the fact that $\lambda = 3$. It will be shown next that the set of solutions II′ cannot lead to an orthogonal array. Consider two triples of rows, namely the triple consisting of the second, third, and fourth rows, and that consisting of the second, third, and fifth rows. Since $\lambda = s = t = 3$, each of these triples of rows has to include three columns of each of the following four types.

The column has a zero in the fourth or fifth column, respectively, and one of the four possible couples consisting of one's and two's only in the second and third rows. By II′ $n^5_{13} = n^5_{11} = 0$. Thus each of the last twelve columns of the first third of the orthogonal array will have a zero either in the fourth or in the fifth row but not in both. Hence these twelve columns will be divided into two groups each consisting of six columns of the considered types such that one group belongs to the triple of rows including the second, third, and fourth rows, and the other to the triple containing the second, third, and fifth rows. These groups are clearly not identical in respect to their content—one's and two's—in the second and third rows. The remaining six columns of the considered types will have to be among the last twelve columns of the second and third parts of the array. Since $n^5_{11} = 0$, these six columns will have to be identical regarding their content in the second and third row. This is clearly impossible.

Finally, the nonexistence of the orthogonal array satisfying solution III′ or IV′ follows immediately from Lemma 1. Clearly, if we delete from an array satisfying III′ one row with a zero in the column having four coincidences with the first column, we will obtain a four-rowed subarray satisfying solution I. In the case of solution IV′ any four-rowed subarray would have to satisfy solution I. This establishes the theorem.

COROLLARY. *Any orthogonal array* (81, $k$, 3, 3) *satisfies the equalities* $n^k_{ij} = 0$, *provided that* $j \geqq 5$ *and* $1 \leqq i \leqq 81$.

THEOREM 2. *The number of constraints of an orthogonal array* (81, $k$, 3, 3) *cannot exceed* 10.

PROOF. Theorem 2 is an immediate consequence of Theorem 1 and Lemma 1 of [2] which established that if, for some $i$, $n_{ij}^{10} = 0$ for all $j \geqq 5$, then such an array cannot be extended to an eleven-rowed orthogonal array.

REMARK. It was also shown in Lemma 1 of [2] that if $k = 10$, then the array satisfies a unique set of solutions. Namely, $n_{i4}^{10} = 60$, $n_{i3}^{10} = n_{i2}^{10} = 0$, $n_{i1}^{10} = 20$, $n_{i0}^{10} = 0$, for all $i = 1, 2, \cdots, 81$. Hence any array constructed by the geometrical method developed by Bose and Bush [1] will satisfy this set of solutions. The problem of obtaining the totality of orthogonal arrays was investigated neither in the considered case nor in related cases.

In conclusion, we wish to remark that this paper restores the validity of the abstract published in *Ann. Math. Stat.*, Vol. 25 (1954), p. 177, which was unduly corrected in [2].

REFERENCES

[1] R. C. BOSE AND K. A. BUSH, "Orthogonal arrays of strengths two and three," *Ann. Math. Stat.* Vol. 23, (1952), pp. 508-524.

[2] E. SEIDEN, "On the maximum number of constraints of an orthogonal array," *Ann. Math. Stat.*, Vol. 26, (1955), pp. 132-135.

# A THEOREM ON CONVEX SETS WITH APPLICATIONS[1]

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**1. Summary and introduction.** T. W. Anderson [1] has proved the following theorem and has given applications to probability and statistics.

THEOREM 1. *Let $E$ be a convex set in $n$-space, symmetric about the origin. Let $f(x) \geqq 0$ be a function such that* i) $f(x) = f(-x)$, ii) $\{x \mid f(x) \geqq u\} = K_u$ *is convex for every $u$ $(0 \leqq u \leqq \infty)$ and* iii) $\int_E f(x)\,dx < \infty$, *then*

$$\int_E f(x + ky)\,dx \geqq \int_E f(x + y)\,dx \qquad \text{for} \quad 0 \leqq k \leqq 1. \tag{1}$$

The purpose of this paper is to prove what can be considered a generalization of Anderson's Theorem and to give different statistical applications.

Functions in $L_1$ satisfying the hypothesis were called unimodal by Anderson and he noted in [1] that if we let $\varphi(y)$ be equal to the right hand side of (1) then $\varphi$ is not unimodal in his sense insofar as it does not necessarily satisfy *ii* (i.e., there exist $f$, $E$, and $u$ such that $\{x \mid \varphi(x) \geqq u\}$ is not convex). His example is

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