Let (Y, Y') designate a random variable having a bivariate normal distribution with means 0, variances 1, and correlation ρ_{12}. Then the limiting distribution of $a(S_{n1} - E\{S_{n1}\}) + b(S_{n2} - E\{S_{n2}\})$ is the distribution of $aY + bY'$ for all a, b. If we knew that $(S_{n1} - E\{S_{n1}\}, S_{n2} - E\{S_{n2}\})$ had a limiting distribution, say the distribution of a random variable (Z, Z'), then it would follow that the linear compound would have the distribution of $aZ + bZ'$. But this means that the random variable $aY + bY'$ is equivalent to $aZ + bZ'$ for all a, b. By Cramér [5], p. 105, this implies that the random variables (Z, Z') and (Y, Y') are equivalent. If a limiting distribution did not exist for

$$(S_{n1} - E\{S_{n1}\}, S_{n2} - E\{S_{n2}\}),$$

then we could extract on n two subsequences which have limiting distributions that are different. This contradicts the statement that the limiting distribution must be that of (Y, Y'). This proves the limiting normality when the correlation approaches a limit.

If $\lim_{n\to\infty} \rho_{n12}$ does not exist, then we can extract on n two subsequences with different limits. Then, by the argument above, the two subsequences of random variables would have limiting normal distributions which are different. This implies that the original sequence of random variables does not have a limiting distribution. The proof is completed.

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ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Princeton meeting of the Institute, April 20–21, 1956)

1. On Certain Stabilities of Sample Survey Response, (Preliminary Report), David Rosenblatt, American University, (By Title).

In economic and demographic surveys, studies of reporting behavior are sometimes undertaken through reinterviews of identical respondents using a similar or more intense mode of inquiry. Stability of response is examined in the light of cross-tabulation data on first vs. second response. Assume: (1) there exist response observables $\alpha_1, \cdots, \alpha_r$, cryptostates $\beta_1, \cdots, \beta_L$, generally unobservable, which may, but need not, correspond to response observables; (2) $L \times r$ stochastic matrices Φ_1, Φ_2, respectively, giving for each crypto-state the conditional probabilities of response in first and second surveys; (3) an initial distribution h over crypto-states for some set of respondents; (4) an $L \times L$ stochastic matrix Q governing transitions among crypto-states for respondents in interval between surveys. In effect, the conditional distributions of second response depend directly only on second crypto-state and, in turn, the probabilities of entering second crypto-states depend directly only on first crypto-states. The expected joint distribution of first and second response is then given by $C(h) = \Phi_1' D(h) Q \Phi_2$, where diagonal matrix $D(h)$ has h on main diagonal. A n.a.s.c. for the marginal distributions of $C(h)$ to be equal, given any h, is that $\Phi_1 = Q\Phi_2$; any $C(h)$ is then necessarily symmetric. If one assumes crypto-states to be a fixed reference frame for historical truth, $L = r$, and $\Phi_2 = I$, the identity, then an observable cross-tabulation may provide an estimate of response pattern Φ_1. (Received March 12, 1956).

2. **Some Multi-Level Continuous Sampling Plans,** C. DERMAN, S. LITTAUER, and H. SOLOMON, Columbia University.

Lieberman and Solomon (*Ann. Math. Stat.*, Vol. 26(1955), pp. 686–705) introduced a multilevel continuous sampling plan which allows for any number of sampling levels subject to the provision that transitions can occur only between adjacent levels. Some extensions of these plans are discussed. Specifically, (i) the situation where transition to levels having smaller sampling rates occurs one level at a time when quality is good and immediate transition to 100 per cent inspection occurs at any sampling level when quality is poor; (ii) the situation where transition to levels having smaller sampling rates occurs s levels at a time when quality is good and transition to levels having higher sampling rates occurs h levels at a time when quality is poor. The Average Outgoing Quality Function (AOQ) is derived for (i) for k levels, finite and infinite. For k infinite the Average Outgoing Quality Limit (AOQL) is derived and seems to have a simple relationship to the contours of equal AOQL already exhibited in the Lieberman-Solomon paper. For $k = 1$, the AOQ reduces to that of the Dodge plan (*Ann. Math. Stat.*, Vol. 14(1943), pp. 264–279). For (ii), bounds are obtained for the AOQL. (Received March 12, 1956.)

3. **Applications of Vector-Valued Risk Functionals, (Preliminary Report),** ALLAN BIRNBAUM, Columbia University.

Lionel Weiss has shown (*Ann. Math. Stat.*, Vol. 24(1953), pp. 677–80) that the methods of decision theory have natural extensions to the case in which the real-valued loss function is replaced by a vector, of which each component measures one aspect of the desirability of an outcome of a statistical decision problem. Adopting Weiss's notation, a useful generalization is obtained by replacing the risk-component $\gamma_{ik} = \int_Z (\Sigma_{j=1}^{L} \eta_j(x) W_{ijk}(x) dF_i$ by any linear functional $R_u(\eta)$ of the decision function $\eta = (\eta_1(x), \cdots, \eta_L(x))$, and γ^S by the (vector-valued) *linear risk functional* $R(\eta) = (R_1(\eta), \cdots, R_U(\eta))$. If the class of allowed decision functions $\Phi = \{\eta\}$ is convex ($\lambda\eta^{(1)} + (1 - \lambda)\eta^{(2)} \varepsilon \Phi$ if $\eta^{(i)} \varepsilon \Phi, 0 < \lambda < 1$), $R(\eta)$ maps Φ onto a convex U-dimensional set S in which admissible points and complete-class subsets can be characterized in the usual ways. One application is a formal unification of Wald's decision theory with the test theory based on a general form of the Neyman-Pearson lemma. This lemma is proved equivalent to admissibility of a point in S under general conditions. Let $\eta_1(x)$ = probability of rejectiong $H_0: \theta = \theta_0$ when x is observed,

$$\theta = (\theta_1, \cdots \theta_t), R_\theta(\eta) = \int_Z \eta_1(x)\, dF_\theta(x), \qquad R_i(\eta) = \frac{\partial}{\partial\theta_i} R_\theta(\eta)\bigg|_{\theta_0},$$

$$R_{ij} = -\frac{\partial^2}{\partial\theta_i\partial\theta_j} R_\theta(\eta)\bigg|_{\theta_0}, \qquad i, j = 1, \cdots, t.$$

Then $R(\eta) = (R_{\theta_0}, R_1, R_{11})$ leads to tests of Type A (taking $t = 1$), and (taking $t = 2$) $R(\eta) = (R_{\theta_0}, R_1, R_2, R_{12}, R_{11}, R_{22})$ leads to tests of Type C. Other applications (for which Weiss's formulation suffices) are made to mixed single-sample tests and double-sampling tests. (Received March 12, 1956).

4. On Mixed Single-Sample Tests, LEONARD COHEN, Columbia University (introduced by A. Birnbaum).

Let X denote a random variable with cumulative distribution function $F(x, \theta)$, and consider tests of a simple hypothesis $H_0:\theta = \theta_0$ against the simple alternative $H_1:\theta = \theta_1$. A single-sample (fixed sample-size) test is identified with a triple (α, β, n) where α, β denote the probabilities of a type I and type II error, respectively, and n the sample size. A mixed single-sample test is a sequence of quadruples $\{(\gamma_i, \alpha_i, \beta_i, n_i)\}$, where $\gamma_i > 0$, $\sum_0^\infty \gamma_i = 1$, and (α_i, β_i, n_i) is a single-sample test, for $i = 0, 1, 2, \cdots$; γ_i is interpreted as the probability of using the single-sample test $(\alpha_i, \beta_i n_i)$. A mixed single-sample test will be identified with a triple $(\alpha, \beta, n) = \sum_0^\infty \gamma_i(\alpha_i, \beta_i, n_i)$. A mixed single-sample test (α, β, n) is admissible if for any other mixed single-sample test (α', β', n') either $\alpha' > \alpha$, $\beta' > \beta$ or $n' > n$. Treating (α, β, n) as a vector-valued risk function, a method for constructing a complete class of admissible mixed single-sample tests is developed and applied to tests on parameters of normal, binomial, and rectangular distributions. For each testing problem it is shown which fixed sample-size tests can be improved upon and by how much, by use of mixed tests. Examples of such improvements have been given by Kruskal and Raiffa. The method is also applied to confidence intervals evaluated in terms of average sample sizes, average lengths and confidence coefficients. (Received March 12, 1956.)

5. On the Use of Randomization in the Investigation of an Unknown Function, ROBERT HOOKE, Westinghouse Electric Corporation.

Much of research in the physical sciences consists in the experimental study of a function of one or more variables. Experimental errors of a random kind are usually present, but often there are biases which are of equal or greater importance. In some cases of this sort the bias can be removed, as in the classical theory of design of experiments, by randomization. For example, if it is desired to estimate a definite integral, randomization implies taking observations at randomly selected (rather than equally spaced) points in the range of integration. Stratified sampling can be used, and a proper design can provide an unbiased estimate of the integral, togeher with estimates of error broken down into a part attributable to experimental error and a part attributable to the random sampling process. Similar observations may apply to the goodness-of-fit problem. (Received March 12, 1956.)

6. Factorials in Near-Balanced Incomplete Block Designs for $k(k - 1)$ Treatments, (Preliminary Report), C. Y. KRAMER and R. A. BRADLEY, Virginia Polytechnic Institute, (By Title).

In this paper the adjusted treatment sum of squares in near-balanced incomplete block designs with $k(k - 1)$ treatments is given as a function of the estimated treatment effects. This new form is valid both when the blocks are grouped into replications in the form of the near-balanced rectangular lattice or the latinized rectangular lattice and when no grouping into replications is effected. By imposing factorial arrangements of treatments on these designs and by using orthogonal contrasts, the sums of squares for the factorial effects with one degree of freedom are found as functions of the estimated treatment effects. A complete analysis of variance for the factorials set in these designs is developed. If the factorial treatments are assigned to the blocks in a special manner, the sums of squares for a $(k - 1)$-level factor are obtained free of block adjustment. This work is preliminary to a more intensive study of the analysis of factorial arrangements of treatments introduced into incomplete block designs of various classes. That this can be done for any balanced incomplete block designs seems to be well known but not discussed in statistical literature. (Received March 12, 1956.)

7. The Comparative Importance of the Order of an Observation in Determining Linear Estimates of the Mean and Standard Deviation of the Normal Distribution from Censored Samples, $N < 10$, A. E. SARHAN and B. G. GREENBERG, University of North Carolina.

Tables were calculated for linear estimation of the mean and standard deviation in censored samples $\leqq 10$ from a normal population in previous work. The coefficients and relative efficiencies in these tables present certain patterns which shed light on the relative contributions made by observations dependent upon order in the sample. For example, for the known sample elements after censoring, the relative contribution which undergoes maximum change in estimating the mean and standard deviation is that associated with the extreme known sample elements. Also, for a fixed sample size and fixed number of censored elements from the left, the importance of the largest known element increases whereas that of the smallest known element decreases as the number of censored elements on the right increases.

The table for relative efficiencies shows that the efficiencies of the estimate of σ drop more rapidly than that of μ. This table also shows that for fixed n and fixed uncensored sample size, the efficiency of the best estimate of σ is constant independent of censored elements from the right or left.

Furthermore, results show that, in estimating the mean, a single central value is worth more than half the sample, while this is the reverse in estimating σ. (Received March 12, 1956.)

8. Relations Between Stochastic Processes, BAYARD RANKIN, Massachusetts Institute of Technology.

For each $t \geqq 0$, $X(t)$ and $Y(t)$ are random variables defined on $(\Omega, \mathfrak{A}, P)$. $\mathfrak{B}(X(t))$ is the Borel field induced on Ω by $X(t)$ and $P^{X(t)}B$ is the conditional probability of B w.r.t. $X(t)$. The separable stochastic processes $X \equiv \{X(t), t \geqq 0\}$, $Y \equiv \{Y(t), t \geqq 0\}$ are related if $\mathfrak{B}(Y) \subset \mathfrak{B}(X)$. (All following statements hold for all $t > \tau \geqq 0$, $B \varepsilon \mathfrak{B}(Y(t))$.) We investigate the special relation defined by (1) $\mathfrak{B}(Y(t)) \subset \mathfrak{B}(X(t))$ and (2) $P^{Y(\tau)}B = P^{X(\tau)}B$. If Y^- is any stochastic process such that $Y^-(t)$ induces the same Borel field as $\{Y(\tau), 0 \leqq \tau \leqq t\}$, then $X = Y^-$ satisfies (1). The following is basic to this paper: THEOREM: *Let X, Y satisfy* (1). $P^{Y(\tau)}B = P^{Y^-(\tau)}B$ *and* $P^{Y^-(\tau)}B = P^{X^-(\tau)}B$ *if and only if* $P^{Y(\tau)}B = P^{X(\tau)}B$ *and* $P^{X(\tau)}B = P^{X^-(\tau)}B$. This theorem has the logical form AB if and only if CD. It is proved that neither A, B, C, nor D imply one another (the theorem is strong) and there exist X, Y satisfying (1) and AB (the theorem is nontrivial). The former follows from the THEOREM: *For each of the conditions $AC\bar{D}$, $AD\bar{C}$, $BC\bar{D}$, $BD\bar{C}$ there exist X, Y satisfying it as well as* (1), *where $\bar{D} \equiv$ not D.* This work was supported in part by the Office of Naval Research. (Received March 12, 1956.)

9. The Two Sample Multivariate Problem in the Degenerate Case, (Preliminary Report), A. P. DEMPSTER, Princeton University.

Samples of n_1 and n_2 individuals are available from 2k-variable normal populations with common variances and covariances but possibly different means. When $n_1 + n_2 - 2 < k$, the classical methods for assessing population separation fail, so this may be called the degenerate case. It appears necessary to give up the desirable property of invariance of the method under all linear transformations of k-space, and so the present method depends on an original choice of units. In k-space vectors V_0 joining the sample means and V_{rj} joining the rth sample mean to the jth point of the rth sample ($r = 1, 2; j = 1, 2, \cdots, n_r$) are defined. By transforming on individuals $n_1 + n_2 - 2$ linear combinations of the V_{rj} can be found which are independently and identically distributed and, under the null hypothesis H of equal means, independent of and distributed as V_0. Deviations from H will show by V_0 being "too long". The distribution theory of vector lengths and other geometrically motivated statistics is described and approximations given. Corresponding approximate tests are discussed, some based only on lengths and others seeking to use more of the information present. (Received March 12, 1956.)

10. A Test for Independence in Contingency Tables, (Preliminary Report), JAMES G. C. TEMPLETON, Princeton University.

It is well known that the Pearson χ^2 test for independence in contingency tables lacks power when used as a test against certain specific alternatives. A test is proposed, based upon "missing-plot" estimates of the expected frequencies, which is believed to be more satisfactory than the χ^2 test when the departure from independence is due to disturbances confined to a few cells of the table. (Received March 12, 1956)

11. An Extension of a Method of Making Multiple Comparisons, (Preliminary Report), THOMAS E. KURTZ, Princeton University.

A method of Tukey [*Proc. Fifth Ann. Convention, Amer. Soc. for Quality Control*, (1951), p. 189.] is extended to include the treatment of data having unequal variances. The allowance for comparisons of the form $(Y_i - Y_j)$, $i, j = 1, 2, \cdots, n$, is $q_\alpha(n, f)s_f\sqrt{(g_i^2 + g_j^2)/2}$, where $\mathrm{Var}(Y_i) = g_i^2\sigma^2$, s_f^2 is the usual unbiased estimate of σ^2, and $q_\alpha(n, f)$ is the αth percent point of the studentized range of n based on f degrees of freedom. A norm for contrasts is obtained such that the resulting statements about contrasts follow as logical consequences from the set of statements about comparisons. The error rate of this extension is shown to be $\leqq \alpha$ for $n = 3$ and for certain special values of the $\{g_i\}$. The implication is that the error rate is always $\leqq \alpha$. The procedure would thus be conservative. When compared with a method of Scheffé [*Biometrika*, Vol. 40 (1953), p. 87.], the extension is shown to provide intervals at least as short for unequal variances as the unmodified Tukey method does for equal variances. (Received March 6, 1956.)

12. On the Power of Some Rank Order Two-Sample Tests, JOAN RAUP ROSENBLATT, National Bureau of Standards.

In a given class $\mathfrak{D}$ of pairs of distributions (F, G), the null hypothesis $F = G$ is to be tested against a wide class of alternatives $\mathfrak{D}_1 \subset \mathfrak{D}$. Among rank-order tests considered are the Mann-Whitney test, a two-sample test proposed by Lehmann (1951), and others. The power of a test is considered as a function of $\theta = \int F\, dG$, $\Delta^2 = \frac{1}{2}\int (F - G)^2\, d(F + G)$, and other functionals of the pair (F, G), under the condition that $\mathfrak{D}_1$ contains only pairs of continuous distributions. Also considered is the symmetrical problem of deciding between two subsets $\mathfrak{D}_0$, $\mathfrak{D}_1$ of $\mathfrak{D}$ when each is defined in terms of values of $\theta(F, G)$. Comparisons among the various tests with respect to power or efficiency are made for several ranges of sample sizes. (Received March 21, 1956).

(Abstracts of papers presented at the Chicago meeting of the Institute, April 27–28, 1956)

13. A Note on Ranking Means, W. A. Thompson, Jr., Fort Bliss, Texas.

Many methods have been proposed for ranking means in analysis of variance and it has been suggested that the method to use may depend on the nature of the loss function. The results of this paper seem to indicate that for one plausible type of loss function the "old fashioned" least significant difference test is appropriate. (Received March 13, 1956).

14. Multivariate Ratio Estimation for Finite Populations, Ingram Olkin, University of Chicago.

In sample surveys, precision in estimating the unknown population mean $\bar{Y}$ may be increased by using an auxiliary variate X, which is positively correlated with Y and whose mean $\bar{X}$ is known. Two such procedures are ratio and regression estimation. This paper is concerned with a multivariate extension of ratio estimation. Let $(Y, X_1, \cdots, X_p)$ denote a finite multivariate population of N elements, where the means $\bar{X}_1, \cdots, \bar{X}_p$ are known. On the basis of a sample $(y_i, x_{1i}, \cdots, x_{pi})$ $i = 1, \cdots, n$, $\bar{Y}$ is to be estimated. The proposed ratio estimate is a linear combination of the individual ratio estimates: $y = w_1 r_1 \bar{X}_1 + \cdots + w_p r_p \bar{X}_p$, where $r_i = y \mid \bar{x}_i$, and $w = (w_1, \cdots, w_p)$ is a weight function. The optimum weight function which minimizes the approximate mean square error is given by

$$w = e(A + b'b)^{-1} \mid e(A + b'b)^{-1}e',$$

where $e = (1, \cdots, 1)$, $b_i = \sqrt{n}(C_i^2 - \rho_{0i}C_0C_i)$ $(i = 1, \cdots, p)$, $A: p \times p$, $a_{ij} = C_0^2 - \rho_{0i}C_0C_i - \rho_{0j}C_0C_j + \rho_{ij}C_iC_j$; C_i is the coefficient of variation of $X_i (Y \equiv x_0)$, ρ_{ij} is the correlation between X_i and X_j.

Estimates of the approximate variance and mean-square error of $\bar{y}$ and some comparisons of estimation procedures are given. Also, an extension to stratified sampling and the approach to normality are considered. (Received March 9, 1956.)

15. Runs Above the Sample Mean, Herbert T. David, University of Chicago.

Let $R_n^{(d)}$ be the number of runs of length d above the sample mean, in a sample of n from a normal population. The distribution of $R_n^{(d)}$ is computed in closed form for n:4, 5. Formulas are given for evaluating the distribution of $R_n^{(d)}$ for all finite n. The evaluation of this distribution amounts essentially to solving a problem (the Demon Problem) posed by J. Youden. Asymptotic upper and lower bounds are also given for $P\{R_n^{(1)} = 1\}$, n large. It is further shown that $R_n^{(d)}$ is asymptotically normal, with asymptotic variance V_d given by the equation $W_d - V_d = (2/\pi)(W_d - U_d)$, where W_d and U_d are, respectively, the asymptotic variances of runs of length d above the population mean and population median, as given by A. M. Mood. (Received March 9, 1956).

16. Approximations to the Power of Rank Tests, Chia Kuei Tsao, Wayne University, (By Title).

The paper proposes a method for approximating the distribution of the ranks, which is the basis for evaluating the power of an arbitrary rank test. Let $F_0, \cdots, F_k$ be $k + 1$ continuous cdf's. Let $Z_{i1}, \cdots, Z_{im_i}$ be the ordered results of a random sample drawn from $F_i(z)$, $i = 0, \cdots, k$. Let $\theta = (\theta_{01}, \cdots, \theta_{0m_0}, \cdots, \theta_{k1}, \cdots, \theta_{km_k})$ be the ranks of

$$Z = (Z_{01}, \cdots, Z_{0m_0}, \cdots, Z_{k1}, \cdots, Z_{km_k})$$

Let $T(z)$ be a continuous, strictly increasing function such that $T(-\infty) = 0$, $T(\infty) = 1$. Let $\Phi_i(v) = \Phi_i(T(z)) = F_i(z)$, $i = 0, \cdots, k$. Let $V_{i1}, \cdots, V_{im_i}$ be the ordered results of a random sample drawn from the cdf $\Phi_i(v)$, $i = 0, \cdots, k$. Let

$$\gamma = (\gamma_{01}, \cdots, \gamma_{0m_0}, \cdots, \gamma_{k1}, \cdots, \gamma_{km_k})$$

be the ranks of $V = (V_{01}, \cdots, V_{0m_0}, \cdots, V_{k1}, \cdots, V_{km_k})$. Then the distribution of θ, say $P(R)$, is identical with that of γ, and hence, according to a throem due to Hoeffding [*Proceedings of the Second Berkeley Symposium on Math. Stat. and Prob.*, 1951, p. 88], it can be expressed as a multiple integral of essentially the product of $\Phi_i'(v)$, the derivatives of $\Phi_i(v)$. Since, for many known cdf's $F_i(z)$, the functions $\Phi_i(v)$ are of unknown form, the author proposes the use of interpolating polynomials $Q_i(v)$ as approximations to $\Phi_i(v)$. Consequently, approximate values of $P(R)$ can be obtained by elementary integrations when $\Phi_i'(v)$ are replaced by $Q_i'(v)$, the derivatives of $Q_i(v)$. For the case $k = 1$, explicit formulae for approximating $P(R)$ by this method are given. As illustrations, a few tables are calculated for the normal alternatives. The same approximation procedure may be used to obtain the large sample power of certain rank tests using large sample theory. The asymptotic power efficiency of a class of rank tests is also investigated. (Received March 9, 1956).

17. On Maximizing and Minimizing a Certain Integral with Statistical Applications, JAGDISH S. RUSTAGI, Carnegie Institute of Technology.

The problem considered in this paper is that of minimizing and maximizing $\int_{-X}^{X} \phi\,(x, F(x))\,dx$ under the assumptions that $F(x)$ is a cumulative distribution function (cdf) on $[-X, X]$ with the first two moments given and ϕ is a certain known function having certain properties. The existence of the solution has been proved and a characterization of the minimizing and maximizing cdf's given. The minimizing cdf is unique when $\phi(x, y)$ is strictly convex in y and is completely characterized for some special forms of ϕ. However, the maximizing cdf is a discrete distribution and in the above case turns out to be a three-point distribution. Using a technique due to Karlin, the minimum problem has been reduced to that of minimizing $\int_{-X}^{X} [\partial\phi(x, F_0(x))/\partial y + \eta_1 + \eta_2 x]\,F(x)\,dx$ over the class of all cdf's on $[-X, X]$, where $F_0(x)$ is the minimizing cdf and η_1, η_2 are constants. This is a problem linear in $F(x)$ and simpler to deal with. An interesting result proved is that

$$\{x{:}\partial\phi(x, F_0(x))/\partial y + \eta_1 + \eta_2 x,\ -X < x < X\}$$

has F_0-measure zero. Results of Gumbel and of David and Hartley (*Ann. Math. Stat.*, 1954) have been obtained as special cases of the above problem and some other interesting statistical applications are discussed. (Received March 9, 1956.)

18. Relations Between Stochastic Variables, GERHARD TINTNER, Iowa State College.

Information theory permits a unified treatment of various types of multivariate analysis (Kullback). It is possible to integrate weighted regression and the limited information method into this approach. A related method is the method of Theil. Other methods of obtaining estimates of relations between stochastic variables are: Instrumental variables, minimum distance methods, nonparametric methods. The problem of identification (uniqueness(is discussed and also difficulties arising because of autocorrelation and serial correlation. (Received April 19, 1956).

19. On the Distribution of the Likelihood Ratio, Robert V. Hogg, State University of Iowa.

Consider the k populations having probability density functions $Q(\theta_i)P(x)$, $a(\theta_i) \leqq x \leqq b(\theta_i)$, zero elsewhere, $i = 1, 2, \cdots, k$, where $P(x)$ is a real single-valued positive continuous function of x and either (1): $a(\theta)$ equals a constant and $b(\theta)$ is a strictly monotone continuous function; or (2): $a(\theta)$ equals θ and $b(\theta)$ is a strictly monotone decreasing continuous function. We test, by use of the likelihood ratio λ, the hypothesis that

$$(\theta_1, \theta_2, \cdots, \theta_k)$$

belongs to a certain type of m-dimensional subset of the k-dimensional parameter space against all possible alternatives. Under the null hypothesis it is proved that $-2 \ln \lambda$ has an exact χ^2 distribution with $2(k - m)$ degrees of freedom. (Received March 13, 1956).

20. Quadratic Time Homogeneous Birth and Death Processes, (Preliminary Report), Peter W. M. John, University of New Mexico, (By Title).

Time homogeneous birth and death processes, in which $\lambda_n = \lambda(n^2 + an)$ and

$$\mu_n = \mu(n^2 + an),$$

where λ, μ, a are constants with $a \geqq 0$, and $n(0) = 1$, are considered. A necessary and sufficient condition that such a process shall be divergent, in the sense that there is a positive probability of an infinite number of events occurring in finite time, it seen to be that $\lambda > \mu$. The differential equations for the probability generating function and the cumulant generating function for $n(t)$ are obtained. On equating coefficients the latter equation yields a series of differential equations, which can be solved for the individual cumulants if, and only if, the process is a balanced one, i.e., $\lambda = \mu$. In this case the moments of $n(t)$ and of the total population count $M(t)$ are easily calculated. (Received March 21, 1956).

21. Seasonal Forecast of Some Time Series, (Preliminary Report), Joseph V. Talacko, Marquette University.

Stationary time series with the predominant seasonal fluctuation, with time observations (like reported cases of infectious diseases, etc.): $y_{11}, y_{12}, \cdots, y_{1r}; y_{21}, \cdots, y_{2r}; \cdots y_{ij} \cdots y_{nr}$, where n is a number of years and r a number of seasonal equidistant intervals, offer a reasonable forecasting of the most probable total observations, with arbitrary confidence intervals soon in the season. For sufficiently large n, the average cumulative relative frequencies may be graduated and the graduation function $G(t)$ used as the cumulative probability function. The seasonal random variations from the $G(t)$ obey the Poisson law as a function of the density $g(t)$, so the confidence intervals of the seasonal forecast may be found for any $0 < t < 2\pi$ from a simple recurrent formula. The Fournier or Logit regression may be applied with some advantages. Application on the morbidity data of the poliomyelitis in the United States, based on weekly and monthly observations since 1920, is introduced. (Received March 21, 1956).

22. Confidence Intervals for Variance Ratios Specifying Genetic Heritability, Franklin A. Graybill, Oklahoma A and M College.

Consider the twofold analysis of variance model with equal subclass numbers, $y_{ijk} = \mu + a_i + b_{ij} + c_{ijk}$, where y_{ijk} is the observation, μ is a fixed constant, and the a_i, b_{ij}, and c_{ijk} are independent normal variables whose means are zero and whose variances are σ_a^2, σ_b^2, and σ_c^2 respectively.

In some fields of endeavor (especially in genetics) the above model occurs quite frequently, and an estimate of the quantity, $h^2 = 2(\sigma_a^2 + \sigma_b^2)/\sigma_a^2 + \sigma_b^2 + \sigma_c^2$, is ofttimes desired. In this paper a method is presented for obtaining an approximate confidence interval for h^2 which is quite accurate even for very small sample sizes. The method is patterned somewhat after the method used by Satterthwaite to obtain the approximate distribution of linear functions of Chi-square variates. (Received March 30, 1956).

23. **A Method of Multiple Rank Correlation,** H. THEIL, University of Chicago and Netherlands School of Economics (introduced by D. L. Wallace).

Consider three rankings x_{0i}, x_{1i}, x_{2i} $(i = 1, \cdots, n)$, all permutations of the first n integers, the latter two being fixed. The null hypothesis is that the x_0-ranking is independent of the other two, with equal probabilities for all $n!$ possibilites. For this purpose the following coefficient of multiple rank correlation is proposed: $T_{0.12} = (t_{01} + t_{02})/(1 + \tau_{12})$, t_{01} being the Kendall rank correlation of x_0 and x_1, τ_{12} the (fixed) rank correlation of x_1 and x_2, etc. $T_{0.12}$ is obtained by scoring according to Kendall for the two rankings x_0 and x_1, and for the rankings x_0 and x_2, adding the scores, and dividing by the attainable maximum, given the fixed x_1- and x_2-rankings.

We have $-1 \leqq T_{0.12} \leqq 1$ and, under the null-hypothesis, $ET_{0.12} = 0$ and var $T_{0.12} = (\frac{4}{9})\binom{n}{2}^{-1}\{(n+1)(1+\rho_{12}^S) + (\frac{2}{3})(1+\tau_{12})\}/\{1+\tau_{12}\}^2$, ρ_{12}^S being the Spearman rank correlation for x_1, x_2. The distrubution of $T_{0.12}$ under the null hypothesis is asymptotically normal.

The further analysis includes a discussion of the relation between $T_{0.12}$ and Kendall's coefficient of partial rank correlation, a discussion of Moran's coefficient of multiple rank correlation, and a generalisation for more variables. (Received March 30, 1956).

24. **Moments of a Test Criterion for Outliers,** P. A. KEYS, Iowa State College.

We assume a random sample, $x_1, \cdots, x_n$, of size n from $N(0, 1)$ arranged in ascending order of magnitude so that x_1 is the smallest and x_n the largest observation. It is sometimes of interest to test whether x_n is in fact the largest observation in a random sample or whether it ought to be discarded as an outlier. Thompson, *Ann. Math. Stat.*, Vol. 6 (1935) pp. 214–219, and Pearson and Chandra Sekar, *Biometrika*, Vol. 28 (1936), pp. 308–320, have considered as a test criterion the ratio $u = (x_n - \bar{x})/s$ where s^2 is the sample mean square, $s^2 = \Sigma(x_i - \bar{x})/n - 1)$. By a special argument Pearson and Chandra Sekar were able to derive the extreme upper percentage points of u. Formulas for the moments of u have been derived in this paper. Since u is distributed independently of s, the moments of u can be derived from those of $(x_n - \bar{x})$ and s. A specially adapted characteristic function technique, previously used by McKay, *Biometrika*, Vol. 27 (1935), pp. 466–471, was employed to obtain formulas for the moments of $(x_n - \bar{x})$. McKay's technique links the moments of $(x_n - \bar{x})$ to those of x_n. The latter have recently been tabulated by Ruben, *Biometrika*, Vol. 41 (1954), pp. 200–227. The derivation and tabulation of these moments of the extreme deviate permits an approximation to the distribution by a Pearson type fit to the first four exact moments.

25. **Post Stratification in Multistage Sampling,** WILLIAM H. WILLIAMS, Iowa State College.

Let ${}_iM$ denote the number of units of a population falling into the ith stratum of a number (L) of strata $(i = 1, 2, \cdots L)$ and ${}_iY$ denote the total of the characteristics ${}_iy_t(t = 1, 2, \cdots {}_iM)$. Their mean is ${}_i\bar{Y} = {}_iY/{}_iM$.

In certain situations one cannot determine the exact stratum to which a unit belongs until after it has been sampled. In these situations a device known as post stratification is sometimes used, consisting of the following steps—(a) Draw a random sample of size m. (b)

Denote by ${}_i\bar{y}$ the mean of the ${}_im$ units which happen to be sampled in the ith stratum. (c) Estimate the population total by $\hat{Y} = \sum_{i=1}^{L} {}_iM\,{}_i\bar{y}'$ where ${}_i\bar{y}' = {}_i\bar{y}$ if ${}_im > 0$ and ${}_i\bar{y}' = 0$ if ${}_im = 0$. The approximate variance formula for this estimator can be found in the literature. Here a new approach is used to derive these formulas. This method, which uses the ratio estimator ${}_i\hat{Y}$ of the strata mean ${}_i\bar{Y}$ can be generalized to cover post stratification in any sampling design however complex. The result is as follows: Let $\hat{Y}({}_iy_t)$ denote the estimate of the total for the particular sampling scheme which has been used and $V({}_iy_t)$ denote its variance. Then the estimate of the population total is $\sum_{i=1}^{L} {}_iM\,{}_i\hat{y}$ and its variance is $V({}_iy_t')$ where ${}_iy_t' = {}_iy_t - {}_i\bar{Y}$ if ${}_iy_t$ is in the ith stratum.

This approach permits consideration of different systems of post stratification with regard to precision. (Received April 20, 1956).

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of interest

Personal Items

The Degree of Doctor of Laws was conferred on Harold Hotelling by the University of Chicago on 11 November, 1955, at the convocation in celebration of the twenty-fifth anniversary of the University's Social Science Research Building. Dr. Hotelling was cited as the "foremost contemporary contributor of quantitative methods to the social sciences, who by mathematical analysis has notably advanced our understanding of fundamental problems in economics and in statistics."

George E. Auman has been appointed Assistant Chief of the National Bureau of Standards Management Planning Division, where he will assist in Bureau's management program.

Paul M. Blunk has left Operations Research Group, Convair, Fort Worth, Texas, to become Chief, Reliability Group, Aerojet General Corp., Sacramento, Calif. He will apply principles of probability and mathematical statistics to the field of rocket reliability.

Derrill J. Bordelon received an M.A. in Mathematics from the University of Maryland, February 1, 1956.

William Fuller Brown, Jr., formerly of the Physical Laboratory of the Sun Oil Company, Newtown Square, Pa., has accepted a position as a Senior Physicist in the Central Research Department of Minnesota Mining and Manufacturing Company, St. Paul, Minnesota.

Jack Chassan has accepted the position of Chief Statistician at Saint Elizabeth's Hospital in Washington, D. C.

Dr. Willard H. Clatworthy has accepted a position as Staff Statistician with Westinghouse Electric Corporation, Atomic Power Division, Bettis Plant, Pittsburgh 30, Pa.

Mr. John L. Dalke has been appointed chief of the High Frequency Imped-