can be evaluated by interpolation using equation (20) and Table I of [9]. However, the number of decimals in Table I of [9] is not sufficiently large to yield accurate enough calculated values of $\Phi_n(u)$.

REFERENCES

[1] R. C. BOSE, AND J. ST-PIERRE, "On a decision procedure to select the population with the largest mean", Abstract, *Ann. Math. Stat.*, Vol. 25 (1954), pp. 813–814.

[2] J. O. IRWIN, "The further theory of Francis Galton's individual difference problem," *Biometrika*, Vol. 17 (1925), pp. 100–128.

[3] J. ST-PIERRE, "Distribution of linear contrasts of order statistics," *Institute of Statistics*, University of North Carolina, Chapel Hill, July, 1954.

[4] S. S. WILKS, *Mathematical Statistics*, Princeton University Press, Princeton, N. J., 1943.

[5] *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments*, National Bureau of Standards, 1939.

[6] *Tables of the Exponential Function* e^x, Applied Mathematics Series, XIV, National Bureau of Standards, 1951.

[7] *Tables of Normal Probability Functions*, Applied Mathematics Series, XXIII, National Bureau of Standards, 1953.

[8] C. JORDAN, *Calculus of Finite Differences*, Chelsea Publishing Company, New York, 1947.

[9] R. E. BECHHOFER, "A single-sample multiple decision procedure for ranking means of normal populations with known variances," *Ann. Math. Stat.*, Vol. 25 (1945), pp. 16–40.

A CERTAIN CLASS OF SOLUTIONS TO A MOMENT PROBLEM

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1. Summary. A uniqueness and a characterization theorem are given for the density function over the interval $[-1, 1]$ with a given finite sequence of moments whose square has the smallest possible integral. Extensions are indicated.

2. Existence and characterization theorems.

Let $\mu_0 = 1, \mu_2, \cdots, \mu_n$ be a given set of real numbers ($0 \leqq n < \infty$). Necessary and sufficient conditions on $(\mu_0, \cdots, \mu_n)$ that there be at least one density function $f(x)$ over $[-1, 1]$ with

$$\int_{-1}^{1} x^i f(x)\, dx = \mu_i, \qquad i = 0, \cdots, n, \tag{1.1}$$

$$\int_{-1}^{1} f^2(x)\, dx < \infty$$

have been given [1]. Throughout this paper, we shall assume that the sequence $(\mu_0, \cdots, \mu_n)$ satisfies these conditions. Then we have:

THEOREM 1. *Let $\{f\}$ denote the class of density functions over $[-1, 1]$ satisfying (1.1), and let M denote*

$$\underset{f(x) \text{ in } \{f\}}{\text{g.l.b.}} \int_{-1}^{1} f^2(x)\, dx.$$

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There is a function $g(x)$ in $\{f\}$ with $\int_{-1}^{1} g^2(x)\,dx = M$. Any function in $\{f\}$ with this property equals $g(x)$ almost everywhere.

PROOF. We can find a sequence $f_1(x), f_2(x), \cdots$ of functions in $\{f\}$ with $\int f_i^2$ approaching M as i increases. Let ϵ_i denote $\int f_i^2 - M$. Then $\int [f_i - f_j]^2 = 2M + \epsilon_i + \epsilon_j - 2\int f_i \cdot f_j \geqq 0$, so $\int f_i \cdot f_j \leqq M + \frac{1}{2}(\epsilon_i + \epsilon_j)$. Also, $\frac{1}{2}(f_i + f_j)$ is in $\{f\}$, so that $\int [\frac{1}{2}(f_i + f_j)]^2 = \frac{1}{4}(M + \epsilon_i) + \frac{1}{4}(M + \epsilon_j) + \frac{1}{2}\int f_i f_j \geqq M$, or $\int f_i \cdot f_j \geqq M - \frac{1}{2}(\epsilon_i + \epsilon_j)$. Thus, $\int f_i f_j$ approaches M as $1/i + 1/j$ approaches zero, and therefore $\int [f_i - f_j]^2$ approaches zero with $1/i + 1/j$. But then it is known ([2], p. 243) that there is a measurable function $g(x)$ such that $\int [f_i - g]^2$ approaches zero as i increases. But $g(x)$ is in $\{f\}$, for it must be non-negative almost everywhere on $(-1, 1)$, and

$$\left| \int_{-1}^{1} x^j g(x)\,dx - \mu_j \right| = \left| \int_{-1}^{1} x^j [g(x) - f_i(x)]\,dx \right|$$
$$\leqq \left(\int_{-1}^{1} x^{2j}\,dx \right)^{1/2} \left(\int_{-1}^{1} [g - f_i]^2\,dx \right)^{1/2},$$

the term on the right approaching zero as i increases. Also, $\int_{-1}^{1} g^2(x)\,dx = M$, for $\int g^2 = \int f_i^2 + \int (g - f_i)^2 + 2\int f_i(g - f_i)$, and as i increases the expression on the right of this last equality approaches M. If a function $f(x)$ in $\{f\}$ has $\int_{-1}^{1} f^2(x)\,dx = M$, then $f(x) = g(x)$ almost everywhere. For $\frac{1}{2}(f + g)$ is in $\{f\}$; therefore $\int [\frac{1}{2}(f + g)]^2 = \frac{1}{2}M + \frac{1}{2}\int fg \geqq M$, or $\int fg \geqq M$. But if f fails to equal g on a set of positive measure, $\int fg < (\int f^2)^{1/2}(\int g^2)^{1/2} = M$, a contradiction. Therefore $g(x)$ is essentially unique.

THEOREM 2. *The function $g(x)$ described in Theorem 1 is, almost everywhere on $(-1, 1)$, equal to a certain polynomial $P(x)$ of degree at most n wherever $P(x)$ is non-negative, and is equal to zero elsewhere.*

PROOF. Denote the polynomial of degree i in the sequence of polynomials orthonormal on a given bounded set S of positive measure by $Q(x, i, S)$, so $\int_S Q(x, i, S)Q(x, j, S)\,dx = \delta_{ij}$ (the Kronecker delta). Since the sequence $\{Q(x, i, S)\}$ is complete in the class of functions whose squares are Lebesgue integrable over S, a necessary and sufficient condition that a function $r(x)$ in this class is, almost everywhere on S, equal to a polynomial of degree at most n, is that $\int_S r(x)Q(x, i, S)\,dx = 0$ for all $i > n$. Also, of all functions $s(x)$ whose squares are Lebesgue integrable and which have $\int_S s(x)x^i\,dx = c_i$, $i = 0, \cdots, n$, by Parseval's Theorem ([2], p. 251) one with the smallest $\int_S s^2(x)\,dx$ is the polynomial $v(x)$ of degree at most n uniquely determined by $\int_S v(x)x^i\,dx = c_i$, $i = 0, \cdots, n$. For any positive ϵ, let G_ϵ denote the subset of $(-1, 1)$ where $g(x) \geqq \epsilon$. Assume ϵ is small enough so that the measure of G_ϵ (written $\mathrm{m}(G_\epsilon)$) is positive. Then, almost everywhere on G_ϵ, $g(x)$ must be equal to a certain polynomial of degree at most n, say $P(x)$ ($P(x)$ will not depend on ϵ). For if not, there is an $i > n$ so that $\int_{G_\epsilon} g(x)Q(x, i, G_\epsilon)\,dx \neq 0$. Then we can find a positive δ so that $g(x) + \gamma Q(x, i, G_\epsilon)$ is positive on G_ϵ for all γ with $|\gamma| < \delta$. But a γ_0 with $0 < |\gamma_0| < \delta$ can be found so that

$$\int_{G_\epsilon} [g(x) + \gamma_0 Q(x, i, G_\epsilon)]^2\,dx = \int_{G_\epsilon} g^2 + 2\gamma_0 \int_{G_\epsilon} gQ + \gamma_0^2 < \int_{G_\epsilon} g^2.$$

But then if we define $h(x)$ as equal to $g(x) + \gamma_0 \cdot Q(x, i, G_\epsilon)$ on G_ϵ, and equal to $g(x)$ elsewhere, $h(x)$ is in $\{f\}$, and $\int_{-1}^{1} h^2 < \int_{-1}^{1} g^2$, a contradiction. Therefore, almost everywhere on G_ϵ, $g(x) = P(x)$. Now we take a sequence of decreasing positive numbers converging to zero, say $\epsilon_1, \epsilon_2, \cdots$. Let R_{ϵ_i} be the subset of G_{ϵ_i} where $g(x)$ fails to equal $P(x)$. Then $R_{\epsilon_1}, R_{\epsilon_2}, \cdots$ is a nondecreasing sequence of sets, and $m(R_{\epsilon_i}) = 0$ for all i. Now $R_{\epsilon_1} + R_{\epsilon_2} + \cdots$ is the set where $g(x) \neq 0$ and $g(x) \neq P(x)$, and $m(R_{\epsilon_1} + R_{\epsilon_2} + \cdots) = \lim_{i\to\infty} m(R_{\epsilon_i}) = 0$. Therefore, almost everywhere where $g(x)$ does not equal zero, $g(x)$ equals $P(x)$. Now let P_ϵ be the subset of $(-1, 1)$ where $P(x) \geqq \epsilon$. Almost all points of G_ϵ are in P_ϵ. Suppose $0 < m(G_\epsilon) < m(P_\epsilon)$. Then, denoting the complement of G_ϵ by $\bar{G}_\epsilon$, we can adjoin to G_ϵ a subset of $\bar{G}_\epsilon \cdot P_\epsilon$ of positive measure, to get a set G'_ϵ. The polynomial $q(x)$ of degree at most n defined by $\int_{G'_\epsilon} q(x)x^i\,dx = \int_{G'_\epsilon} g(x)x^i\,dx$, $i = 0, \cdots, n$, must be negative somewhere on G'_ϵ, for if not we could decrease $\int_{-1}^{1} g^2$ by replacing it by $q(x)$ on G'_ϵ ($g(x)$ cannot equal $q(x)$ almost everywhere on G'_ϵ, for $g(x) = P(x)$ on G_ϵ, zero on $G'_\epsilon - G_\epsilon$). But the polynomial defined on G_ϵ as $q(x)$ is defined on G'_ϵ is at least ϵ everywhere on G_ϵ, and by making $m(G'_\epsilon)$ close enough to $m(G_\epsilon)$, we can make certain that $q(x)$ is non-negative on G'_ϵ, for the coefficients of $q(x)$ vary continuously as $m(G'_\epsilon)$ grows. This contradiction proves that $m(G_\epsilon) = m(P_\epsilon)$. Taking a sequence $\epsilon_1, \epsilon_2, \cdots$ as above, the set G where $g(x)$ is positive is $G_{\epsilon_1} + G_{\epsilon_2} + \cdots$, the set P where $P(x)$ is positive is $P_{\epsilon_1} + P_{\epsilon_2} + \cdots$. Then $m(G) = \lim m(G_{\epsilon_i}) = \lim m(P_{\epsilon_i}) = m(P)$, so $m(G) = m(P)$. Since almost every point of G is in P, we have that almost everywhere where $P(x)$ is positive $g(x) = P(x)$.

3. Extensions. The results above can be generalized as follows.

THEOREM 3. *Given a bounded set S of positive measure, and measurable functions $h_0(x), h_1(x), \cdots, h_n(x)$ such that $\int_S h_i^2(x)\,dx$ is finite for $i = 0, \cdots, n$, and numbers $m_0, m_1, \cdots, m_n$, suppose that there is at least one measurable function $f(x)$ with the following properties*:

(a) $f(x) \geqq 0$ *almost everywhere on* S,

(b) $\int_S f(x)h_i(x)\,dx = m_i$, $i = 0, \cdots, n$,

(c) $\int_S f^2(x)\,dx$ *is finite.*

Then there is a measurable function $g(x)$ with these properties, uniquely defined almost everywhere on S, such that $\int_S g^2(x)\,dx$ achieves the g. l. b. *of $\int_S f^2(x)\,dx$ taken over the class of functions with properties* (a), (b), *and* (c). *Further, $g(x)$ is equal to a certain linear combination $L(x)$ of $h_0(x), \cdots, h_n(x)$ wherever $L(x)$ is non-negative, and $g(x)$ is equal to zero wherever $L(x)$ is negative.*

PROOF. Exactly the same as in Section 2, except that the orthonormal sequence of functions starts with linear combinations of $h_0(x), \cdots, h_n(x)$ instead of linear combinations of $x^0, \cdots, x^n$.

REFERENCES

[1] N. I. ACHIESER AND M. KREIN, *On Certain Problems in the Theory of Moments*, Kharkov, 1938.

[2] L. M. GRAVES, *The Theory of Functions of Real Variables*, McGraw-Hill, New York, 1946.