and ρ_1 is always zero. Also,

$$V(\hat{\rho}_j) = \sigma_1^2 \Big/ \left(r - \frac{1}{k} d_j\right), \qquad j = 2, 3, \cdots, t,$$

where σ_1^2 is the within-block variance for the incomplete block design, so

$$\frac{1}{t(t-1)} E\left[\sum_{\substack{i,j \\ i \neq j}} (\hat{\tau}_i - \hat{\tau}_j)^2 \Big/ \text{all } \tau_j = 0\right] = \frac{2\sigma_1^2}{t-1} \sum_{j=2}^{t} \frac{1}{\left(r - \frac{d_j}{k}\right)},$$

which is the mean variance of a treatment difference.

For the complete block design, the mean variance of a treatment difference is $2\sigma_2^2/r$, where σ_2^2 is the variance within blocks for the complete block design.

Hence, we have the final result. The efficiency factor (EF) of an incomplete block design is equal to r times the harmonic mean of the latent roots of the matrix of coefficients of the reduced normal equations for the intrablock estimates, excluding the always-present zero root, whose characteristic vector consists of the same number repeated t times.

It may be of interest to record the view point that while the efficiency factor is a reasonable criterion of the loss due to confounding by blocking, from some points of view the generalized variance would be better. This, of course, corresponds in a certain sense to the geometric mean of the latent roots.

4. Notes on the Result. The result is interesting to the author and appears to be worth recording in the literature. It was obtained in a search for a proof of a theorem that the design with the highest efficiency factor is a balanced incomplete block design if such a design exists. To the author's knowledge, this theorem is yet to be proved.

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THE NULL DISTRIBUTION OF THE DIFFERENCE BETWEEN THE TWO LARGEST SAMPLE VALUES[1]

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1. Introduction. A decision procedure to select the population with the largest mean, proposed by Bose and St-Pierre [1], involves the auxiliary statistic $u = x_{(0)} - x_{(1)}$, where $x_{(0)}$ and $x_{(1)}$ are respectively the largest and second largest

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TABLE I

$\Phi_n(u)$

u	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
0.0	.00 000	.00 000	.00 000	.00 000	.00 000	.00 000
0.2	.16 321	.19 439	.21 630	.23 298	.24 623	.25 720
0.4	.31 250	.36 450	.39 967	.42 565	.44 592	.46 239
0.6	.44 573	.50 943	.55 078	.58 040	.60 299	.62 100
0.8	.56 178	.62 964	.67 183	.70 110	.72 289	.73 993
1.0	.66 040	.72 669	.76 607	.79 249	.81 168	.82 639
1.2	.74 217	.80 295	.83 737	.85 968	.87 547	.88 734
1.4	.80 831	.86 126	.88 975	.90 760	.91 985	.92 888
1.6	.86 051	.90 464	.92 718	.94 078	.94 990	.95 648
1.8	.90 067	.93 604	.95 315	.96 307	.96 956	.97 416
2.0	.93 081	.95 815	.97 065	.97 761	.98 204	.98 515
2.2	.95 287	.97 329	.98 208	.98 676	.98 966	.99 168
2.4	.96 861	.98 335	.98 926	.99 223	.99 398	.99 515
2.6	.97 957	.98 990	.99 383	.99 577	.99 688	.99 770

values in a sample of size $n + 1$ taken from a normal population with zero mean and unit variance. The null distribution of u might be obtained from a formula given by Irwin [2], but computations, based on it, seem rather complicated. Another form of the distribution was obtained by St-Pierre [3], involving iterated integrals of the normal density.

2. The null distribution of u**.** Let us denote by $\phi_n(u)$, the p.d.f. of $u = x_{(0)} - x_{(1)}$ in the case of a sample of size $n + 1$. An expression for $\phi_n(u)$, more amenable to calculations than the ones previously mentioned, can be derived using, as a starting point, the joint distribution of the ordered sample values [4]. It has the following form:

$$\phi_n(u) = \frac{(n+1)!e^{-u^2/4}}{(2\pi)^{n/2}\sqrt{2}} \int_{u/\sqrt{6}}^{\infty} e^{-y_2^2/2} \int_{y_2/\sqrt{2}}^{\infty} \cdots \int_{(y_{n-1}\sqrt{n-1})/\sqrt{n+1}}^{\infty} e^{-y_n^2/2}\, dy_n \cdots dy_2 .$$

3. Tabulation of $\Phi_n(u)$**, the c.d.f. of** u**.** Rapidly converging series expansions of $\phi_n(u)$ were derived. With the help of [5], [6], and [7], $\phi_n(u)$ was computed for u: 0.0(0.2)2.6. Finally $\Phi_n(u)$ was obtained by numerical integration methods [8]. Table I gives $\Phi_n(u)$ for $n = 2, 3, \cdots, 7$.

4. Remark. As was pointed out by the referee, $\Phi_n(u)$ can be obtained directly. It can be shown that

$$P[x_{(0)} - x_{(1)} \leqq \lambda] = \Phi_n(\lambda)$$
$$= 1 - (n+1)\left\{\sum_{j=0}^{n} (-1)^j \binom{n}{j} \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \left[\int_{-\infty}^{y+\lambda} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt\right]^j dy\right\}.$$

The terms involving integrals of the type

$$\int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \left[\int_{-\infty}^{y+\lambda} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt\right]^j dy$$

can be evaluated by interpolation using equation (20) and Table I of [9]. However, the number of decimals in Table I of [9] is not sufficiently large to yield accurate enough calculated values of $\Phi_n(u)$.

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A CERTAIN CLASS OF SOLUTIONS TO A MOMENT PROBLEM

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1. Summary. A uniqueness and a characterization theorem are given for the density function over the interval $[-1, 1]$ with a given finite sequence of moments whose square has the smallest possible integral. Extensions are indicated.

2. **Existence and characterization theorems.** Let $\mu_0 = 1, \mu_2, \cdots, \mu_n$ be a given set of real numbers $(0 \leqq n < \infty)$. Necessary and sufficient conditions on $(\mu_0, \cdots, \mu_n)$ that there be at least one density function $f(x)$ over $[-1, 1]$ with

$$\begin{aligned} &\int_{-1}^{1} x^i f(x)\,dx = \mu_i, \qquad i = 0, \cdots, n, \\ &\int_{-1}^{1} f^2(x)\,dx < \infty \end{aligned} \tag{1.1}$$

have been given [1]. Throughout this paper, we shall assume that the sequence $(\mu_0, \cdots, \mu_n)$ satisfies these conditions. Then we have:

Theorem 1. *Let* $\{f\}$ *denote the class of density functions over* $[-1, 1]$ *satisfying* (1.1), *and let* M *denote*

$$\underset{f(x) \text{ in } \{f\}}{\text{g.l.b.}} \int_{-1}^{1} f^2(x)\,dx.$$

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