NOTES

ON THE TUKEY TEST FOR THE EQUALITY OF MEANS AND THE HARTLEY TEST FOR THE EQUALITY OF VARIANCES[1, 2]

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1. Summary. The unbiasedness of the Tukey Studentized range test for the equality of means of k univariate normal populations with a common variance and of the Hartley $F_{\max}$ ratio test for the equality of variances of k univariate normal populations is proved.

2. Introduction. The purpose of this paper is to establish the unbiasedness of two tests which are derived by the union-intersection principle [2], the tests being within the Neyman-Pearson set-up of two-decision problems.

3. The Tukey q-test. Let $x_{ij} (i = 1, 2, \cdots, k; j = 1, 2, \cdots, n)$ be the elements of k independent samples of size n from normal populations with means μ_i and variance σ^2 $(i = 1, 2, \cdots, k)$. Also let s^2 be an independent and unbiased estimate of σ^2 based on m d.f. (say, the error mean square in anova). It is well known that $\bar{x}_i = \sum_{j=1}^n x_{ij}/n$ is normal with mean μ_i and variance σ^2/n.

To test the hypothesis $H_0 : \mu_1 = \mu_2 = \cdots = \mu_k$ we proceed as follows: First we notice that H_0 is equivalent to the totality of all $H_{ij}^0 : \mu_i = \mu_j$ $(i \neq j, i, j = 1, 2, \cdots, k)$. Also for any two μ's, the hypothesis $\mu_i = \mu_j$ can be tested using Student's "t" with m d.f. The hypothesis $\mu_i = \mu_j$ is accepted if $| \bar{x}_i - \bar{x}_j | \leqq t_\gamma s(2/n)^{1/2}$ where t_γ is the upper $\gamma/2$ point of Student's "t" with m d.f. Now since H_0 is equivalent to the totality of the hypothesis $H_{ij}^0 (i \neq j, i, j = 1, 2, \cdots, k)$, we get a test of H_0 as follows: Take the intersection of all the $\binom{k}{2}$ two-by-two Student's "t_{ij}" acceptance regions, and accept H_0 if

$$\text{largest } | t_{ij} | = \sup_{i \neq j, i, j = 1, 2, \cdots, k} \left\{ | \bar{x}_i - \bar{x}_j | / s \sqrt{\frac{2}{n}} \right\} \leqq t_\gamma .$$

It is easy to check that this is the same as accepting H_0 if

$$q = \frac{\bar{x}_{\max} - \bar{x}_{\min}}{s \sqrt{\frac{2}{n}}} \leqq Q,$$

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where Q is the upper α point of the Studentized range q with m d.f. (Notice that $t_\gamma = Q$.) This is the Tukey q-test [3].

Starting with the definition of the q-test, we have, for the probability of the second kind of error,

$$\beta = \Pr\left\{\frac{\bar{x}_{\max} - \bar{x}_{\min}}{s\sqrt{\frac{2}{n}}} \leqq Q\right\} \tag{3.1}$$
$$= \Pr\left\{\frac{y_{\max} - y_{\min}}{s'} \leqq Q\sqrt{2}\right\},$$

where $y_i = \sqrt{n}\, \bar{x}_i/\sigma$ $(i = 1, 2, \cdots, k)$ and $s' = s/\sigma$. Now y_i is normal with mean μ_i' and variance unity, where $\mu_i' = (n/\sigma^2)^{1/2}\mu_i$ $(i = 1, 2, \cdots, k)$. Also, s' has the distribution of $(\chi_m^2/m^2)^{1/2}$ independent of y's.

Now since the test is invariant under location transformations, we have

$$\beta = \sum_1^{k-1} \int_0^\infty p_1(s) \int_{-\infty}^\infty p(z) \int_{z-\eta_i}^{z-\eta_i+Q's} p(t)\, dt \prod_{\substack{j=1\\ j\neq i}}^{k-1} \int_{z-\eta_i+\eta_j}^{z-\eta_i+\eta_j+Q's} p(t)\, dt\, dz\, ds \tag{3.2}$$
$$+ \int_0^\infty p_1(s) \int_{-\infty}^\infty p(z) \prod_{j=1}^{k-1} \int_{z+\eta_j}^{z+\eta_j+Q's} p(t)\, dt\, dz\, ds,$$

where

$$Q' = Q\sqrt{2},$$
$$p(z) = \frac{e^{-z^2/2}}{\sqrt{2\pi}},$$
$$p_1(s) = \text{const}\, s^{m-1} e^{-ms^2/2},$$
$$\eta_{i-1} = \mu_1' - \mu_i' \qquad (i = 2, 3, \cdots, k).$$

From (3.2) it is evident that β involves as parameters only the $k - 1$ η's. Hence the power $(=1 - \beta)$ of the q-test involves as parameters only the $k - 1$ η's. It is worth noting at this point that the right side of (3.2) is symmetric in the η's. Hence the power of the q-test is also symmetric in the η's.

4. Unbiased nature of the q-test. To prove the unbiased nature of the q-test we need to use certain lemmas, which we shall now prove.

LEMMA 1.[4] Suppose that

(1) in the domain D: $(\mathbf{x}: a_i \leqq x_i \leqq b_i,\ i = 1, 2, \cdots, k)$, $f(x_1, x_2, \cdots, x_k)$ exists, all partial derivatives of order one and two exist, all partial derivatives of order one vanish simultaneously at one and only one inner point

$$P = (x_{10}, x_{20}, \cdots, x_{k0}) \text{ of } D;$$

(2) the matrix of second partials evaluated at P is negative definite (n.d.); and

(3) at every point $(x_1, x_2, \cdots, x_k)$ on the boundary of D, $f(x_1, x_2, \cdots, x_k) < A$, where $A = f(x_{10}, x_{20}, \cdots, x_{k0})$.

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[4] The author wishes to thank the referee for suggesting the present proof of Lemma 1.

Then

$$f(x_1, x_2, \cdots, x_k) < A \tag{4.1}$$

for all $x \varepsilon D$, $x \neq P$.

PROOF. Because the domain is closed and the function is continuous, max $f(x_1, x_2, \cdots, x_k) = B$, say, exists. Suppose $B \geqq A$ and that for

$$(x_1^*, x_2^*, \cdots, x_k^*) \neq (x_{10}, x_{20}, \cdots, x_{k0}), \qquad f(x_1^*, x_2^*, \cdots, x_k^*) = B.$$

By Condition 3, $(x_1^*, x_2^*, \cdots, x_k^*)$ is not on the boundary. By Condition 1, at least one partial derivative is not zero at $(x_1^*, x_2^*, \cdots, x_k^*)$, say the derivative with respect to x_1. Suppose it is positive; i.e., that

$$\left.\frac{\partial f(x_1, x_2, \cdots, x_k)}{\partial x_1}\right]_{x=x^*} > 0.$$

Then for sufficiently small δ, $f(x_1^* + \delta, x_2^*, \cdots, x_k^*) > f(x_1^*, x_2^*, \cdots, x_k^*) = B$, which is contrary to the assumption. Hence the lemma.

LEMMA 2. If the conditions of Lemma 1 are satisfied as $a_i \to -\infty$ or $b_i \to \infty$, for any i and for fixed values of a_j, $b_j (j \neq i, j = 1, 2, \cdots, k)$, then $f(x_1, \cdots, x_k) < A$ for all $x \varepsilon D'$: $\{\mathbf{x}: -\infty < x_i < \infty, i = 1, \cdots, k\}H$, $\mathbf{x} \neq P$.

PROOF. The proof follows obviously from Lemma 1.

THEOREM 1. *The Studentized range test of Tukey is unbiased.*

PROOF. Differentiating β with respect to η_1 we get, after some simplification,

$$\begin{aligned}\frac{\partial \beta}{\partial \eta_1} = \int_0^\infty p_1(s) \int_{-\infty}^\infty \{p(z)p(z + \eta_1 + Q's) \\ - p(z + \eta_1)p(z + Q's)\} \prod_{j=2}^{k-1} \int_{z+\eta_j}^{z+\eta_j+Q's} p(t)\, dt\, dz\, ds \\ + \sum_{i=2}^{k-1} \int_0^\infty p_1(s) \int_{-\infty}^\infty \{p(z + \eta_i)p(z + \eta_1 + Q's) \\ - p(z + \eta_1)p(z + \eta_i + Q's)\} \prod_{\substack{j \neq i \\ j=2}}^{k-1} \int_{z+\eta_j}^{z+\eta_j+Q's} p(t)\, dt \int_z^{z+Q's} p(t)\, dt\, dz\, ds.\end{aligned} \tag{4.2}$$

It is easy to check that the right side of (4.2) will be negative if $\eta_1 > 0$ and $\eta_1 > \eta_i (i = 2, 3, \cdots, k - 1)$ and positive if $\eta_1 < 0$ and

$$\eta_1 < \eta_i (i = 2, 3, \cdots, k - 1).$$

By the symmetry in the variables the same is true of $\partial\beta/\partial\eta_i (i = 2, 3, \cdots, k - 1)$; i.e.,

$$\begin{aligned}\frac{\partial \beta}{\partial \eta_i} < 0 \qquad \text{if } \eta_i > 0 \text{ and } \eta_i = \eta_{\max}, \\ \frac{\partial \beta}{\partial \eta_i} > 0 \qquad \text{if } \eta_i < 0 \text{ and } \eta_i = \eta_{\min}.\end{aligned} \tag{4.3}$$

Also it is evident that

(the notation $\eta = 0$ will mean $\eta_1 = \eta_2 = \cdots = \eta_{k-1} = 0$, etc.)

$$\left.\frac{\partial\beta}{\partial\eta_1}\right]_{\eta=0} = 0. \tag{4.4}$$

Similarly,

$$\left.\frac{\partial\beta}{\partial\eta_i}\right]_{\eta=0} = 0 \qquad (i = 2, 3, \cdots, k-1).$$

Now suppose $\eta^0 \neq 0$. Then either $\eta^0_{\max} > 0$ or $\eta^0_{\min} < 0$. Hence the first partials can vanish simultaneously only at $(0, 0, \cdots, 0)$.

Again it is easily verified that

$$\left.\frac{\partial^2\beta}{\partial\eta_i^2}\right]_{\eta=0} = -(k-1)Q'c(Q'), \tag{4.5}$$

where

$$c(Q') = \int_0^\infty sp_1(s) \int_{-\infty}^\infty \exp -\left[\frac{z^2}{2} + \tfrac{1}{2}(z + Q's)^2\right]\left[\int_z^{z+Q's} e^{-t^2/2}\,dt\right]^{k-2} ds\,dz > 0.$$

Hence

$$\left.\frac{\partial^2\beta}{\partial\eta_i^2}\right]_{\eta=0} < 0 \qquad (i = 1, 2, \cdots, k-1).$$

Also

$$\left.\frac{\partial^2\beta}{\partial\eta_i\,\partial\eta_j}\right]_{\eta=0} = Q'c(Q') > 0 \qquad (i \neq j, i, j = 1, 2, \cdots, k-1). \tag{4.6}$$

Hence the matrix of second partials, when $\eta = 0$, is

$$M = \left\| \left.\frac{\partial^2\beta}{\partial\eta_i\,\partial\eta_j}\right]_{\eta=0} \right\|$$

$$= \begin{Vmatrix} -(k-1)f(Q') & f(Q') & \cdots & f(Q') \\ f(Q') & -(k-1)f(Q') & \cdots & f(Q') \\ \cdots & \cdots & \cdots & \cdots \\ f(Q') & f(Q') & \cdots & -(k-1)f(Q') \end{Vmatrix} \tag{4.7}$$

where $f(Q') = Q'c(Q')$ is negative definite.

To complete the theorem it will now suffice if we show that $\beta \to 0$ on each point of the boundary of the domain D: $\{\eta: \epsilon_i \leqq \eta_i \leqq \lambda_i; i = 1, 2, \cdots, k-1\}$ as, say, $\epsilon_1 \to -\infty$ or $\lambda_1 \to \infty$ for fixed values of ϵ_i, $\lambda_i (i = 2, 3, \cdots, k-1)$. Now it is easy to verify that as $\epsilon_1 \to -\infty$, the value of β at each point on the boundary $\to 0$. Similarly, it is easy to verify that as $\lambda_1 \to \infty$, the value of β at each point on the boundary $\to 0$. Also the value of β at the point where η's $= 0$ is $1 - \alpha > 0$. Hence all the conditions given in Lemma 2 are satisfied by the function $\beta(\eta)$.

Hence

$$\beta(\eta) < \beta(0) \qquad \text{for every } \eta \neq 0. \tag{4.8}$$

Hence the Tukey q-test is unbiased.

5. The Hartley $F_{\max}$ ratio test. Let $x_{ij}(i = 1, 2, \cdots, k; j = 1, 2, \cdots, n + 1)$ be the elements of k independent samples of size $(n + 1)$ from normal populations with means μ_i and variances $\sigma_i^2(i = 1, 2, \cdots, k)$. It is well known that $s_i^2 = \sum_{j=1}^{n+1} (x_{ij} - \bar{x}_i)^2/n$, where $\bar{x}_i = \sum_{j=1}^{n+1} x_{ij}/(n + 1)$ is an unbiased estimate of $\sigma_i^2(i = 1, 2, \cdots, k)$. It is also well known that ns_i^2/σ_i^2 is a chi-square variable with n d.f.

The hypothesis $H_0 : \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2$ is equivalent to the totality of hypotheses $H_{ij}^0 : \sigma_i^2 = \sigma_j^2(i \neq j; i, j = 1, 2, \cdots, k)$. Now for any two σ's, the hypothesis $\sigma_i^2 = \sigma_j^2$ can be tested using the variance ratio F of Fisher with d.f. (n, n). The hypothesis $\sigma_i^2 = \sigma_j^2$ is accepted if $1/F'_\gamma \leqq (s_i^2/s_j^2) \leqq F'_\gamma$, where F'_γ is the upper $\gamma/2$ point of Fisher's F with d.f. (n, n). Now since H_0 is equivalent to the totality of the hypotheses $H_{ij}^0(i \neq j; i, j = 1, 2, \cdots, k)$, we get a test of H_0 as follows: Take the intersection of all the $\binom{k}{2}$ Fisher's $F_{ij} = (s_i^2/s_j^2)$ acceptance regions and accept H_0 if

$$\text{largest } F_{ij} = \sup_{i \neq j, i, j = 1, 2, \cdots, k} (s_i^2/s_j^2) \leqq F'_\gamma .$$

It is easy to check that this is the same as accepting H_0 if $F_{\max} = (s_{\max}^2/s_{\min}^2) \leqq F$, where F is the upper α point of the $F_{\max}$ distribution with d.f. (n, n). (Notice that $F'_\gamma = F$.) This is the Hartley $F_{\max}$ ratio test [1].

Starting with the definition of the $F_{\max}$ test, we have, since scale transformations leave the test invariant, the probability of the second kind of error

$$\beta = \sum_{i=1}^{k-1} \int_0^\infty p(u) \int_{u/\eta_i}^{Fu/\eta_i} p(v)\, dv \prod_{\substack{j=1\\ j\neq i}}^{k-1} \int_{u\eta_j/\eta_i}^{Fu\eta_j/\eta_i} p(w)\, dw\, du + \int_0^\infty p(u) \prod_{j=1}^{k-1} \int_{u\eta_j}^{Fu\eta_j} p(v)\, dv\, du, \tag{5.1}$$

where

$$p(u) = \text{const } u^{(n/2)-1} e^{-u/2} \qquad \text{and} \qquad \eta_{i-1} = \frac{\sigma_1^2}{\sigma_i^2} \qquad (i = 2, 3, \cdots, k).$$

From (5.1) it is evident that β involves as parameters only the $k - 1$ η's. Hence the power $(= 1 - \beta)$ of the test involves as parameters only the $k - 1$ η's. It is worth noting at this point that the right side of (5.1) is symmetric in the η's. Hence the power of the test is also symmetric in the η's.

6. Unbiased nature of the $F_{\max}$ test. To prove the unbiased nature of the $F_{\max}$ test we need to use a lemma which is

LEMMA 3. If the conditions of Lemma 1 are satisfied as $a_i \to 0$ or $b_i \to \infty$ for any i and for fixed values of $a_j, b_j (j \neq i; j = 1, 2, \cdots, k), f(x_1, \cdots, x_k) < A$ for all $x \in D'$: $\{\mathbf{x}: 0 < x_i < \infty, i = 1, \cdots, k\}, \mathbf{x} \neq P$.

PROOF. The proof follows obviously from Lemma 1.

THEOREM 2. *The $F_{\max}$ test of Hartley is unbiased.*

PROOF. Differentiating β with respect to η_1 we get, after some simplification.

$$\Gamma^2\left(\frac{n}{2}\right)\frac{\partial\beta}{\partial\eta_1} = \int_0^\infty \{u^{n-1}e^{-u(1+F\eta_1)} - u^{n-1}e^{-u(F+\eta_1)}\} \prod_{j=2}^{k-1} \int_{u\eta_j}^{Fu\eta_j} p(v)\,dv\,du$$
$$(6.1) \qquad + \sum_{i=2}^{k-1} \int_0^\infty \{u^{n-1}e^{-u(\eta_i+F\eta_1)} - u^{n-1}e^{-u(\eta_1+F\eta_i)}\}$$
$$\cdot \prod_{\substack{j\neq i\\ j=2}}^{k-1} \int_{u\eta_j}^{Fu\eta_j} p(v)\,dv \int_u^{Fu} p(w)\,dw\,du,$$

where

$$p(v) = \text{const } v^{(n/2)-1}e^{-v}.$$

It is easy to check that the right side of (6.1) will be negative if $\eta_1 > 1$ and $\eta_1 > \eta_i (i = 2, 3, \cdots, k-1)$ and positive if $\eta_1 < 1$ and $\eta_1 < \eta_i (i = 2, 3, \cdots, k-1)$. By the symmetry in the variables, the same is true of $\partial\beta/\partial\eta_i (i = 2, 3, \cdots, k-1)$; i.e.,

$$(6.2) \qquad \begin{aligned} \frac{\partial\beta}{\partial\eta_i} &< 0 \quad \text{if} \quad \eta_i > 1 \quad \text{and} \quad \eta_i = \eta_{\max}, \\ \frac{\partial\beta}{\partial\eta_i} &> 0 \quad \text{if} \quad \eta_i < 1 \quad \text{and} \quad \eta_i = \eta_{\min}. \end{aligned}$$

Also it is evident that

$$(6.3) \qquad \left.\frac{\partial\beta}{\partial\eta_1}\right]_{\eta=1} = 0.$$

Similarly,

$$\left.\frac{\partial\beta}{\partial\eta_i}\right]_{\eta=1} = 0 \qquad (i = 2, 3, \cdots, k-1)$$

Now suppose $\overset{0}{\eta} \neq 1$. Then either $\overset{0}{\eta}_{\max} > 1$ or $\overset{0}{\eta}_{\min} < 1$. Hence the first partials can vanish simultaneously only at $(1, 1, \cdots, 1)$. Again it is easily verified that

$$(6.4) \qquad \left.\frac{\partial^2\beta}{\partial\eta_i^2}\right]_{\eta=1} = (k-1)(1-F)c(F),$$

where

$$c(F) = \frac{1}{\Gamma^k\left(\frac{n}{2}\right)} \int_0^\infty u^n e^{-u} \left[\int_u^{Fu} v^{(n-2)/2}e^{-v}\,dv\right]^{k-2} du > 0.$$

Hence

$$\frac{\partial^2 \beta}{\partial \eta_i^2}\bigg]_{\eta=1} < 0 \qquad (i = 1, 2, \cdots k - 1). \tag{6.5}$$

Also

$$\frac{\partial^2 \beta}{\partial \eta_i \, \partial \eta_j}\bigg]_{\eta=1} = (F - 1)c(F) > 0 \qquad (i \neq j, i, j = 1, 2, \cdots, k - 1). \tag{6.6}$$

Hence the matrix of second partials, when $\eta = \mathbf{1}$, is

$$M = \left\| \frac{\partial^2 \beta}{\partial \eta_i \, \partial \eta_j}\bigg]_{\eta=1} \right\| = \left\| \begin{matrix} -(k-1)g(F) & g(F) \cdots g(F) \\ g(F) & -(k-1)g(F) \cdots g(F) \\ \cdots\cdots & \cdots\cdots \\ g(F) & g(F) \cdots -(k-1)g(F) \end{matrix} \right\|, \tag{6.7}$$

where $g(F) = (F - 1)c(F)$ is negative definite.

To complete the theorem it will now suffice if we show that $\beta \to 0$ on each point of the boundary of the domain $D:\{\eta:\epsilon_i \leqq \eta_i \leqq \lambda_i\,; i = 1, \cdots, k - 1\}$ as, say, $\epsilon_1 \to 0$ or $\lambda_1 \to \infty$ for fixed values of ϵ_i, $\lambda_i (i = 2, 3, \cdots, k - 1)$. It is easy to verify that as $\epsilon_1 \to 0$ the value of β at each point on the boundary $\to 0$. Similarly, it is easy to verify that as $\lambda_1 \to \infty$, the value of β at each point on the boundary $\to 0$. Also the value of β at the point where η's $= 1$ is $1 - \alpha > 0$. Hence all the conditions given in Lemma 3 are satisfied by the function $\beta(\eta)$. Hence

$$\beta(\eta) < \beta(\mathbf{0}) \qquad \text{for every } \eta \neq \mathbf{1}. \tag{6.8}$$

Hence the Hartley F_{max} test is unbiased.

7. Conclusion. So far we considered the F_{max} test when all the s_i^2's are based on the same number of d.f. n. Investigation is proceeding on the behaviour of the F_{max} test when the d.f. are unequal. Power properties of similar generalizations of the q-test are also being investigated.

By inverting the test procedures considered in Sections 3 and 5 useful simultaneous confidence bounds on all two by two differences of the means and all two by two ratios of the variances can be obtained.

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