MATRIX METHODS IN COMPONENTS OF VARIANCE AND COVARIANCE ANALYSIS

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Summary. The sampling variance of the least squares estimates of the components of variance in an unbalanced (non-orthogonal) one-way classification and the large sample variances of the maximum likelihood estimates of these quantities are summarized in a paper by Crump [1]. The present paper outlines a method of obtaining these results by the use of matrix algebra, and extends them to the sampling variances of estimates of components of covariance when two variables are considered. The methods are also used to obtain the large sample variance-covariance matrix of the maximum likelihood estimates of the components of variance and covariance.

Part I. Components of Variance

1. Model and analysis of variance. We are concerned with data in a 1-way classification with unequal numbers of observations in the classes. The linear model is taken as

$$x_{ij} = \mu + \alpha_i + \epsilon_{ij},$$

where x_{ij} is the jth observation in the ith class. We will assume that there are c such classes $(i = 1, \cdots, c)$, the ith class containing n_i observations $(j = 1, \cdots, n_i)$; and let $\sum_i n_i = N$. μ is a general mean, and $\{\alpha_i\}$ and $\{\epsilon_{ij}\}$ are random samples of size c and N from two normally distributed populations having zero means and variances σ_a^2 and σ_e^2, respectively. This is Eisenhart's Model II [3] and it is to this model that the discussion confines itself. The problem is to find the sampling variances of the estimates of σ_a^2 and σ_e^2 based on the usual analysis of variance of between and within classes. These estimates are

$$\hat{\sigma}_e^2 = 1/(N - c)[\sum\sum x_{ij}^2 - \sum n_i\bar{x}_{i.}^2], \tag{1}$$

$$\hat{\sigma}_a^2 = 1/f[1/(c - 1)(\sum n_i\bar{x}_{i.}^2 - N\bar{x}_{..}^2) - \hat{\sigma}_e^2], \tag{2}$$

where $f = 1/(c - 1)(N - \sum n_i^2/N)$, and $\bar{x}_{i.} = 1/n_i\sum_j x_{ij}$, and $\bar{x}_{..} = 1/N\sum_{ij} x_{ij}$.

2. Normal theory. In general if $x_1 \cdots x_N$ is a set of multivariate normally distributed random variables with variance-covariance matrix V, and vector of means zero, their distribution function is given by

$$dH(x_1 \cdots x_N) = \frac{1}{(2\pi)^{N/2} \mid V \mid^{\frac{1}{2}}} \exp \{-\tfrac{1}{2}\mathbf{x}'V^{-1}\mathbf{x}\}\, dx_1 \cdots dx_N,$$

where $\mathbf{x}'$ is the row vector $(x_1 \cdots x_N)$.

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If y is a function of the x's, $y = \mathbf{x}'F\mathbf{x}$, defined by the symmetric matrix F, then the characteristic function of y (with parameter t) is

$$\iint \cdots \int e^{iyt}\, dH(x_1 \cdots x_N),$$

and by the use of Aitken's Integral [7], this can be shown to be equal to $| I - 2itVF |^{-\frac{1}{2}}$, where I is a unit matrix. Then, if $K_r^{(y)}$ is the rth cumulant of y,

$$\sum_r K_r^{(y)} \frac{(it)^r}{r!} = -\tfrac{1}{2} \log |I - 2itVF|.$$

By making use of the properties of the eigenvalues of a symmetric matrix, it has been shown ([9], p. 40; [10], p. 131; and [6], p. 247) that

$$K_r^{(y)} = 2^{r-1}(r-1)!\ \text{trace}\ (VF)^r.$$

For $r = 2$, this gives the result

(3) $$\text{variance}\ (y) = 2\ \text{trace}\ (VF)^2.$$

It is this principle that is used to find the sampling variance of $\hat{\sigma}_e^2$ and $\hat{\sigma}_a^2$ by expressing them in the form $\mathbf{x}'F\mathbf{x}$.

3. Sampling variances of the least squares estimates.

NOTATION. It will simplify procedure if we write a for σ_a^2 and e for σ_e^2 and similarly $\hat{a}$ and $\hat{e}$ for the estimates.

Let the row vector of observations, $(x_{11} \cdots x_{1n_1} \cdots x_{c1} \cdots x_{cn_c})$, be written as $\mathbf{x}'$. Then arraying the data in the order of $\mathbf{x}'$ it is seen that V is a square matrix of order N, the only non-zero elements being c square sub-matrices of order n_i $(i = 1, \cdots, c)$, lying along the diagonal, each with diagonal terms $a + e$ and non-diagonal terms a. Matrices of this particular form we will call A matrices, A_i being defined in general as a square matrix of order n_i, with a_i in all its diagonal terms and b_i everywhere else. The matrix of order N whose only non-zero sub-matrices are A_i's in the diagonal will be termed a C matrix. Thus V is a C matrix, with $a_i = a + e$, and $b_i = a$.

The quadratic form for $(N - c)\hat{e}$ can now be expressed as

$$(N - c)\hat{e} = \mathbf{x}'F_1\mathbf{x},$$

where F_1 is a C matrix with $a_i = 1 - 1/n_i$ and $b_i = -1/n_i$. Thus from (3)

(4) $$(N - c)^2\ \text{var}\ (\hat{e}) = 2\ \text{trace}\ (VF_1)^2,$$

where V and F_1 are C matrices. Now the product of two C matrices is itself a C matrix, and

(5) $$\text{trace}\ C^2 = \sum n_i[a_i^2 + (n_i - 1)b_i^2].$$

Combining these results leads to the well-known expression

(6) $$\text{var}\ (\hat{e}) = \frac{2e^2}{N - c}.$$

The variance of $\hat{a}$ is arrived at in a similar fashion, in the course of which two further matrix types arise. The first we will call a K matrix, K_{ij} being a matrix of order $n_i \times n_j$ with k_{ij} in all its terms. The second is termed a J matrix, a square matrix of order N, being a C matrix with the zero sub-matrices replaced by K_{ij}'s.

In terms of these matrices one can show that the quadratic form of (2) can be expressed as

$$f\hat{a} = \mathbf{x}'F_2\mathbf{x}, \tag{7}$$

where F_2 is a J matrix with

$$a_i = \frac{(N-1)(1/n_i - c/N)}{(c-1)(N-c)},$$

$$b_i = a_i + 1/(N-c),$$

and

$$k_{ij} = -1/N(c-1).$$

Thus VF_2 is the product of a C and a J matrix, which can be shown to be a J matrix with k_{ij} independent of j. For such a matrix

$$\text{trace } (J^2) = \sum n_i[a_i^2 + (n_i - 1)b_i^2] + \left(\sum n_i k_i\right)^2 - \sum n_i^2 k_i^2 . \tag{8}$$

Using these results 2 trace $(VF_2)^2$ is obtainable, thus giving var $(\hat{a})$, which can be written as

$$\text{var } (\hat{a}) = \frac{1}{f^2}\left[\frac{2e^2(N-1)}{(c-1)(N-c)} + \frac{2ea(N^2 - S_2)}{N(c-1)^2} + \frac{2a^2(N^2S_2 + S_2^2 - 2NS_3)}{N^2(c-1)^2}\right], \tag{9}$$

where $S_2 = \sum n_i^2$, and $S_3 = \sum n_i^3$. This is the result given in Crump [2].

It is also of interest to find the sampling variance of the estimate of the total variance, $(\hat{a} + \hat{e})$. By these methods it can be shown that the (VF) matrix for the expression $(c-1)(\hat{e} + f\hat{a})$—i.e., for $\left(\sum n_i \bar{x}_i^2 - N\bar{x}_{..}^2\right)$—is

$$\begin{pmatrix} K_{11} & \cdots & K_{1c} \\ K_{c1} & \cdots & K_{cc} \end{pmatrix},$$

with $k_{ii} = (e + n_i a)(1/n_i - 1/N)$, and $k_{ij} = -1/N(e + n_i a)$. This leads to the result

$$\text{covariance } (\hat{a}, \hat{e}) = (-1/f) \text{ var } (\hat{e}),$$

which gives

$$\text{variance } (\hat{a} + \hat{e}) = (1 - 2/f) \text{ var } (\hat{e}) + \text{var } (\hat{a}). \tag{10}$$

4. Large sample variance of maximum likelihood estimates. The likelihood of the sample, L, is given by

$$e^L = \left(\frac{1}{2\pi}\right)^{N/2} |V|^{-\frac{1}{2}} \exp - \tfrac{1}{2}\mathbf{x}'V^{-1}\mathbf{x}.$$

Thus

$$L = \text{constant} - \tfrac{1}{2}\log|V| - \tfrac{1}{2}\mathbf{x}'V^{-1}\mathbf{x}.$$

Now V is a C matrix with $a_i = a + e$, and $b_i = a$; and it is easily shown that the inverse of a C matrix is a C matrix with terms A_i^{-1}, A_i^{-1} itself being an A matrix. Also

$$|C| = \prod_i |A_i| = \prod_i (a_i - b_i)^{n_i-1}[a_i + (n_i - 1)b_i].$$

These results can be applied to the expression for L, which is then readily differentiable with respect to a and e. Then the inverse of the matrix whose terms are minus the expected values of the second order partial derivatives of L with respect to a and e gives the large sampling variances and covariance of the maximum likelihood estimates of the variance components. These results, due to Crump and quoted here for completeness, are (setting $a/e = Q$)

$$\text{(11)}\qquad \begin{aligned} \text{var}(\tilde{e}) &= 2e^2\textstyle\sum w_i^2/D,\\ \text{var}(\tilde{a}) &= 2e^2[N - c + \textstyle\sum w_i^2/n_i^2]/D,\\ \text{cov}(\tilde{a}\tilde{e}) &= (-2e^2\textstyle\sum w_i^2/n_i)/D, \end{aligned}$$

where $w_i = n_ie/(e + n_ia) = n_i/(1 + Qn_i)$, and $D = N\sum w_i^2 - (\sum w_i)^2$. Thus we have established the well-known results for the least squares estimates of the components of variance, the sampling variances and covariance of these estimates, and the large sampling variances of the maximum likelihood estimates. We now proceed to find the same results for the components of covariance.

Part II. Components of Covariance

5. Least squares estimation. We consider the problem of the components of covariance between two variables x and y, each based on the same linear model in a 1-way classification, under the assumptions of Eisenhart's Model II. a', e' and a'', e'' are taken as the variance components of y, and the covariance components between x and y, respectively, following directly from the notation of paragraph 3.

The least squares estimates of a'' and e'' obtained from the Analysis of Covariance are the same functions of the sums of products of x and y as $\hat{a}$ and $\hat{e}$ were of the sums of squares in the Analysis of Variance:

$$\text{(12)}\qquad \begin{aligned} \hat{e}'' &= 1/(N - c)[\textstyle\sum\sum x_{ij}y_{ij} - \sum n_i\bar{x}_{i.}\bar{y}_{i.}],\\ f\hat{a}'' &= 1/(c - 1)[\textstyle\sum n_i\bar{x}_{i.}\bar{y}_{i.} - N\bar{x}_{..}\bar{y}_{..}] - \hat{e}''. \end{aligned}$$

To find the variance of these estimates we use the same methods as in finding the variance of $\hat{a}$ and $\hat{e}$, namely expressing $\hat{a}''$ and $\hat{e}''$ in the form $\mathbf{x}'F\mathbf{x}$, and, using a variance-covariance matrix V, evaluate 2 trace $(VF)^2$. In this case we are concerned with a random sample of $2N$ variables $(x_{11} \cdots\cdots x_{cn_c}, y_{11} \cdots\cdots y_{cn_c})$ which we assume to be multivariate normally distributed with variance-covariance matrix V_1, say. A little consideration will show that V_1, associated with the vector $(\mathbf{x}'\mathbf{y}')$, is

$$V_1 = \begin{pmatrix} V & V'' \\ V'' & V' \end{pmatrix},$$

where V' and V'' are the same C matrices as V, but in terms of a', e' and a'', e'' respectively. This notation will not be confused with the usual use of primes to denote transpose matrices, since no transposed matrix enters into this analysis.

We now proceed to find the matrix expressions for the sums of products. Writing $\mathbf{z}' = (\mathbf{x}', \mathbf{y}')$ the following results hold:

$$\sum\sum x_{ij}y_{ij} = \tfrac{1}{2}\mathbf{z}'\begin{pmatrix} \cdot & I \\ I & \cdot \end{pmatrix}\mathbf{z},$$

$$\sum n_i\bar{x}_{i.}\bar{y}_{i.} = \tfrac{1}{2}\mathbf{z}'\begin{pmatrix} \cdot & C \\ C & \cdot \end{pmatrix}\mathbf{z}, \qquad \text{with } a_i = b_i = 1/n_i,$$

and

$$N\bar{x}_{..}\bar{y}_{..} = \tfrac{1}{2}\mathbf{z}'\begin{pmatrix} \cdot & K_N \\ K_N & \cdot \end{pmatrix}\mathbf{z}, \qquad K_N \text{ being an } N\times N\ K \text{ matrix with terms } 1/N.$$

These expressions give

$$(N - c)\hat{e}'' = \tfrac{1}{2}\mathbf{z}'\begin{pmatrix} \cdot & F_1 \\ F_1 & \cdot \end{pmatrix}\mathbf{z},$$

and thus the VF matrix for $(N - c)\hat{e}''$ is

$$\tfrac{1}{2}\begin{pmatrix} V & V'' \\ V'' & V' \end{pmatrix}\begin{pmatrix} \cdot & F_1 \\ F_1 & \cdot \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} V''F_1 & VF_1 \\ V'F_1 & V''F_1 \end{pmatrix}.$$

Now each of the four sub-matrices in this expression is the same VF_1 as used for obtaining var $(\hat{e})$ in (4). Therefore in terms of the general result (3), trace $(VF)^2$ for $(N - c)\hat{e}''$ comes from a double application of (5), namely

$$\sum n_i[a_i''^2 + (n_i - 1)b_i''^2 + a_ia_i' + (n_i - 1)b_ib_i'], \tag{13}$$

which leads to the result

$$\text{var}\,(\hat{e}'') = \frac{e''^2 + ee'}{N - c}.$$

A similar procedure holds for var $(\hat{a}'')$. From (12) $f\hat{a}''$ can be written as

$$f\hat{a}'' = \tfrac{1}{2}\mathbf{z}'\left[\frac{N-1}{(N-c)(c-1)}\begin{pmatrix} \cdot & C \\ C & \cdot \end{pmatrix} - \frac{1}{c-1}\begin{pmatrix} \cdot & K_N \\ K_N & \cdot \end{pmatrix} - \frac{1}{N-c}\begin{pmatrix} \cdot & I \\ I & \cdot \end{pmatrix}\right]\mathbf{z}$$

$$= \tfrac{1}{2}\mathbf{z}'\begin{pmatrix} \cdot & F_2 \\ F_2 & \cdot \end{pmatrix}\mathbf{z},$$

where F_2 is the J matrix defined in (7). Therefore

$$\text{var}\,(f\hat{a}'') = 2\,\text{trace}\left[\tfrac{1}{2}\begin{pmatrix} V & V'' \\ V'' & V' \end{pmatrix}\begin{pmatrix} \cdot & F_2 \\ F_2 & \cdot \end{pmatrix}\right]^2 = \tfrac{1}{2}\,\text{trace}\begin{pmatrix} V''F_2 & VF_2 \\ V'F_2 & V''F_2 \end{pmatrix}^2.$$

The sub-matrices of this expression are the same J matrix as considered in obtaining var $(\hat{a})$. Therefore by a double application of (8) similar to (13), var $(f\hat{a}'')$ is obtained. This leads to the result that var $(\hat{a}'')$ equals

$$\frac{1}{f^2}\left[\frac{(N-1)(e''^2+ee')}{(N-c)(c-1)} + \frac{(N^2-S_2)(2e''a''+e'a+ea')}{N(c-1)^2} + \frac{(N^2S_2+S_2^2-2NS_3)(a''^2+aa')}{N^2(c-1)^2}\right], \tag{14}$$

which is the same expression as var $(\hat{a})$ with $(e''^2 + ee')$, $(2e''a'' + ea' + e'a)$, and $(a''^2 + aa')$ replacing $2e^2$, $2ea$, and $2a^2$ respectively.

Finally it can be shown that equation (10) holds for $\hat{e}''$ and $\hat{a}''$, namely

$$\text{var}\,(\hat{e}'' + \hat{a}'') = (1 - 2/f)\,\text{var}\,(\hat{e}'') + \text{var}\,(\hat{a}''). \tag{15}$$

Thus far we have found the variances of the least squares estimates of the components of covariance. The next step is to have the efficiency of these estimates by finding the large sample variance of the maximum likelihood estimates of e'' and a''.

6. Maximum likelihood estimates—large sample variances.

6.1. L, the likelihood function for the sample of $2N$ observations is given by

$$e^L = \left(\frac{1}{2\pi}\right)^{N/2} |V_1|^{-\frac{1}{2}} \exp\,(-\tfrac{1}{2}\mathbf{z}'V_1^{-1}\mathbf{z}),$$

where

$$V_1 = \begin{pmatrix} V & V'' \\ V'' & V' \end{pmatrix} = \begin{bmatrix} A_1 & & & A_1'' & & \\ & \ddots & & & \ddots & \\ & & A_c & & & A_c'' \\ A_1'' & & & A_1' & & \\ & \ddots & & & \ddots & \\ & & A_c'' & & & A_c' \end{bmatrix},$$

with $a_i = a + e$ and $b_i = a$ (and similarly the primed terms) by the definition of paragraph 3.

We will now consider an orthogonal transformation of $\mathbf{z}$, $\mathbf{w} = T\mathbf{z}$, the variance-covariance matrix appropriate to $\mathbf{w}$ being W. With $TT' = I$, $W = TV_1T'$, $V_1 = T'WT$, and

$$e^L = \left(\frac{1}{2\pi}\right)^{N/2} |W|^{-\frac{1}{2}} \exp\,(-\tfrac{1}{2}\mathbf{w}'W^{-1}\mathbf{w}). \tag{16}$$

The value of T which simplifies V most easily is

$$T = \begin{bmatrix} H_1 & & & & & \\ & \ddots & & & & \\ & & H_c & & & \\ & & & H_1 & & \\ & & & & \ddots & \\ & & & & & H_c \end{bmatrix},$$

where H_i is a matrix of order n_i having terms

$$\begin{aligned} h_{pq}^{(i)} &= 1/\sqrt{n_i} \qquad \text{for } p = 1, \text{ and all } q, \\ &= 0 \qquad \text{for } p > 1, \text{ and } q > p, \\ &= 1/\sqrt{p(p-1)} \qquad \text{for } p > 1, \text{ and } q < p, \\ &= -(p-1)/\sqrt{p(p-1)} \qquad \text{for } p > 1, \text{ and } q = p,\ p, q, = 1 \cdots n_i . \end{aligned}$$

As an example, for $n_i = 4$,

$$H_i = \begin{pmatrix} \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \cdot & \cdot \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \cdot \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{-1}{\sqrt{12}} \end{pmatrix}.$$

For this value of T,

$$W = TV_1T',$$

$$W = \begin{pmatrix} D_1 & & & D_1'' & & \\ & \ddots & & & \ddots & \\ & & D_c & & & D_c'' \\ D_1'' & & & D_1' & & \\ & \ddots & & & \ddots & \\ & & D_c'' & & & D_c' \end{pmatrix}, \tag{17}$$

where D_i is a diagonal matrix of order n_i, the leading term being $(e + n_i a)$, the remaining terms e.

6.2. To obtain W and $\mathbf{w}'W^{-1}\mathbf{w}$ of (16) first observe that an elementary matrix of the form

$$M = \begin{pmatrix} m_1 & \cdot & \cdot & m_1'' & \cdot & \cdot \\ \cdot & m_2 & \cdot & \cdot & m_2'' & \cdot \\ \cdot & \cdot & m_3 & \cdot & \cdot & m_3'' \\ m_1'' & \cdot & \cdot & m_1' & \cdot & \cdot \\ \cdot & m_2'' & \cdot & \cdot & m_2' & \cdot \\ \cdot & \cdot & m_3'' & \cdot & \cdot & m_3' \end{pmatrix}$$

can be written as the product

$$M = \begin{pmatrix} m_1 & \cdot & \cdot & m_1'' & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ m_1'' & \cdot & \cdot & m_1' & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & m_2 & \cdot & \cdot & m_2'' & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & m_2'' & \cdot & \cdot & m_2' & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & m_3 & \cdot & \cdot & m_3'' \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & m_3'' & \cdot & \cdot & m_3' \end{pmatrix}.$$

Immediately this gives

$$| M | = \prod_i (m_i m_i' - m_i''^2).$$

Similarly M^{-1} is itself an M matrix, with m_i, m_i'', and m_i' replaced by $m_i'/(m_i m_i' - m_i''^2)$, $-m_i''/(m_i m_i' - m_i''^2)$, and $m_i/(m_i m_i' - m_i''^2)$ respectively.

6.3. These results can be extended, and applied to W as given in (17).

NOTATION. Write

$$p_i = e + n_i a,$$
$$q = ee' - e''^2,$$
$$r_i = p_i p_i' - p_i''^2 = (e + n_i a)(e' + n_i a') - (e'' + n_i a'')^2.$$

Then

$$| W | = q^{N-c} \prod_i r_i,$$

$$W^{-1} = \begin{pmatrix} D_1' & & & -D_1'' & & \\ & \ddots & & & \ddots & \\ & & D_c' & & & -D_c'' \\ -D_1'' & & & D_1 & & \\ & \ddots & & & \ddots & \\ & & -D_c'' & & & D_c \end{pmatrix}, \tag{18}$$

where D_i is a diagonal matrix of order n_i, with leading term p_i/r_i, and other terms equal to e/q. D_i' and D_i'' have the same form as D but with their numera tors primed.

Now $\mathbf{w}'W^{-1}\mathbf{w} = \mathbf{z}'T'W^{-1}T\mathbf{z}$. Furthermore $V_1 = T'WT$, and therefore, since W^{-1} is of the same form as W, $T'W^{-1}T$ has the same form as V_1: its sub-matrices are A matrices. This being so, notice that for a vector of n_i x's, $\mathbf{x}_i$,

$$\begin{aligned} \mathbf{x}_i' A_i \mathbf{x}_i &= (a_i - b_i) \textstyle\sum_j x_{ij}^2 + b_i x_{i.}^2 \\ &= (a_i - b_i)\left(\sum_j x_{ij}^2 - \frac{x_{i.}^2}{n_i}\right) + \frac{x_{i.}^2}{n_i}[a_i + (n_i - 1)b_i]. \end{aligned} \tag{19}$$

Therefore

$$\begin{aligned} \mathbf{w}'W^{-1}\mathbf{w} &= \mathbf{z}'T'W^{-1}T\mathbf{z} \\ &= \textstyle\sum_i (\mathbf{x}_i'A_i'\mathbf{x}_i + \mathbf{y}_i'A_i\mathbf{y}_i - 2\mathbf{x}_i'A_i''\mathbf{y}_i) \end{aligned} \tag{20}$$

can be expressed as a sum of terms like (19). Now the A matrices in V_1 have $a_i = a + e$ and $b_i = a$. $V_1 = T'WT$, and the D matrices of W have leading terms $e + n_i a$ and other terms e. Therefore, since the D matrices of W^{-1} have leading terms p_i/r_i and other terms e/q, the A matrices of $T'W^{-1}T$ have

$$a_i = [(n_i - 1)e/q + p_i/r_i]/n_i$$

and

$$b_i = (p_i/r_i - e/q)/n_i .$$

This gives

$$a_i - b_i = e/q, \qquad \text{and} \qquad a_i + (n_i - 1)b_i = p_i/r_i \tag{21}$$

Substituting expressions (18) to (21) in (16) gives the likelihood as

$$\begin{aligned} L = &- \tfrac{1}{2}N \log(2\pi) - \tfrac{1}{2}(N - c)\log q \\ &- \tfrac{1}{2}\sum_i \log r_i - \tfrac{1}{2}(e'X + eY - 2e''Z)/q \\ &- \tfrac{1}{2}\sum_i (p_i'X_i + p_iY_i - 2p_i''Z_i)/r_i , \end{aligned} \tag{22}$$

where

$$\begin{cases} X = \sum_{ij} x_{ij}^2 - \sum_i n_i \bar{x}_{i.}^2, & \text{with expected value } (N - c)e, \\ X_i = n_i \bar{x}_{i.}^2, & \text{with expected value } p_i . \end{cases} \tag{23}$$

Y, Y_i and Z, Z_i are similar sums of squares of y and sums of products of x and y respectively, with appropriate expected values.

6.4. To find the large-sample variance of the maximum likelihood estimates of all the six components of variance and covariance together, we require the 6×6 matrix whose terms are the expected values of the second order partial derivatives of L with respect to e, e', e'' and a, a', a''. Call this matrix L_2, and consider the row vector of operators:

$$\boldsymbol{\partial}' = \left(\frac{\partial}{\partial e}\frac{\partial}{\partial e'}\frac{\partial}{\partial e''}\frac{\partial}{\partial a}\frac{\partial}{\partial a'}\frac{\partial}{\partial a''}\right).$$

Then $L_2 = -E\partial\partial'L$. Applying this to (22), L_2 will involve the following terms:

$$(24)\qquad \begin{aligned} \partial\partial' \log q &= \frac{1}{q^2}(q\partial\partial' q - \partial q\, \partial' q), \\ \partial\partial'\left(\frac{e}{q}\right) &= \frac{1}{q}(\partial\partial' e) - \frac{1}{q^2}(\partial e\, \partial' q + e\partial\partial' q) + \frac{1}{q^3}(e\partial q\, \partial' q), \end{aligned}$$

and similar expressions for $\partial\partial' \log (r_i)$ and $\partial\partial'(e/r_i)$. Writing

$$U = \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & -2 \end{pmatrix},$$

the terms in (24) can be written as

$$\begin{aligned} \partial' q &= (e' \; e \; -2e'' \; 0 \; 0 \; 0) &&= \mathrm{s}' \text{ say;} \\ \partial\partial' q &= \begin{pmatrix} U & \cdot \\ \cdot & \cdot \end{pmatrix} &&= S, \text{ say;} \\ \partial' r_i &= (p_i' \; p_i \; -2p_i'' \; 0 \; 0 \; 0) &&= \mathrm{t}_i', \text{ say;} \\ \partial\partial' r_i &= \begin{pmatrix} U & n_i U \\ n_i U & n_i^2 U \end{pmatrix} &&= T_i, \text{ say,} \end{aligned}$$

and also,

$$\begin{aligned} \partial' e &= (1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0), \\ \partial' p_i &= (1 \quad 0 \quad 0 \quad n_i \quad 0 \quad 0), \end{aligned}$$

with similar results for e', e'', p_i', and p_i''. All second order derivatives of the e's and p's are zero.

Using these terms and the expected values indicated in (23) it can be shown after a little reduction that

$$-L_2 = \tfrac{1}{2}(N - c)\frac{1}{q}(\mathrm{ss}' - S) + \tfrac{1}{2}\sum_i \frac{1}{r_i^2}(\mathrm{t}_i \mathrm{t}_i' - T_i).$$

This has now to be inverted to give the variance-covariance matrix of the maximum likelihood estimates. If we define

$$P = \begin{pmatrix} e'^2 & e''^2 & -2e'e'' \\ e''^2 & e^2 & -2ee'' \\ -2e'e'' & -2ee'' & 2ee' + 2e''^2 \end{pmatrix},$$

and similarly P_i in terms of the p_i's, then $-L_2$ can be written as

$$(25)\qquad \tfrac{1}{2}\begin{bmatrix} \dfrac{N-c}{q^2}P + \sum \dfrac{1}{r_i^2}P_i & \sum \dfrac{n_i}{r_i^2}P_i \\ \sum \dfrac{n_i}{r_i^2}P_i & \sum \dfrac{n_i^2}{r_i^2}P_i \end{bmatrix}.$$

6.5. Inversion of the matrix in this form does not seem possible, and so in order to make use of it in applications one must at this stage resort to arithmetical methods, replacing the components by their estimates, computing the matrix as it stands, and then inverting it, either directly or by a method of partitioning [5]. For calculating L_2, 24 terms must be computed; 6 of these are the squares and products of the e's multiplied by $(N - c)/q^2$, and the remaining 18 are the sums of squares and products of the p_i/r_i terms, weighted by 1, n_i, and n_i^2. The computing is facilitated by grouping together at all stages all classes having the same number of observations in each class.

6.6. Due to the symmetric nature of P, the upper right (and since L_2 is symmetric also, the lower left) quadrant of L_2^{-1} is symmetric. This means (for example) that the large sample covariance between the maximum likelihood estimates of a between-classes component of variance of x and the within-classes component of variance of y is the same as that between the between-classes component of variance of y and the within-classes component of variance of x; i.e.,

$$\text{cov}\ (\tilde{a}\tilde{e}') = \text{cov}\ (\tilde{a}'\tilde{e}),$$

and similarly

$$\text{cov}\ (\tilde{a}''\tilde{e}) = \text{cov}\ (\tilde{a}\tilde{e}''),$$

$$\text{cov}\ (\tilde{a}''\tilde{e}') = \text{cov}\ (\tilde{a}'\tilde{e}'').$$

6.7. Where two variables have a bivariate normal distribution with variances σ_1^2 and σ_2^2 and unknown correlation coefficient ρ, it can be shown that the large sample variance of the maximum likelihood estimate of σ_i^2 (where ρ is estimated also) is $2\sigma_i^4/n$, $(i = 1, 2.)$ This is the same result as when the two variables are assumed independent. Generalizing this to the case which we have considered, it can be seen that the values in the inverse of the matrix (25) appropriate to the variance components e, a and e', a' will be the same as the expression (11). The matrix, however, gives further information about the covariance components e'' and a'', and also the large sample covariances among the maximum likelihood estimates of all six parameters.

7. Conclusion. Henderson [4] has shown how components of variance can be estimated from unbalanced data in an n-way classification, and states that sampling variances of such estimates are unknown—this is certainly true for n greater than one. This paper presents a matrix method suitable to finding the variance in this known case, the 1-way classification (under the assumptions of Eisenhart's Model II) with a view to extending it to higher classifications. As a first step the method has been shown to give results for the covariance case in a 1-way classification, and it would seem that the 2-way classification for components of variance can be handled in a similar fashion.

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