ASYMPTOTIC DISTRIBUTIONS OF TWO GOODNESS OF FIT CRITERIA

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1. Results. Let $\{X_1, X_2, \dots\}$ be a stochastic process in which each random variable takes as values only the integers $1, 2, \cdots, s$. To test the null hypothesis that the process is independent and stationary with $P\{X_n = k\} = p_k > 0$, it is natural to form the statistic

$$\sum_{u_1, \cdots, u_\nu = 1}^{s} \frac{(n_{u_1 \cdots u_\nu} - np_{u_1} \cdots p_{u_\nu})^2}{np_{u_1} \cdots p_{u_\nu}}, \tag{1.1}$$

where $n_{u_1 \cdots u_\nu}$ is the number of integers $m \leqq n$ for which $(X_m, \cdots, X_{m+\nu-1})$ is the ν-tuple $(u_1, \cdots, u_\nu)$. In Section 2 we show that under the null hypothesis the distribution function of (1.1) approaches, as $n \to \infty$, the distribution function

$$\mathop{*}_{\lambda=1}^{\nu-1} K_{s^{\nu-1-\lambda}(s-1)^2}(x/\lambda) * K_{s-1}(x/\nu), \tag{1.2}$$

where $K_i(x)$ is the chi-square distribution with i degrees of freedom and the first $*$ denotes iterated convolution in the obvious way. Good [1], using different methods, has obtained this result for the special case in which the p_k are all equal and s is a prime number.

If the p_k are estimated by n_k/n, there results the statistic

$$\sum_{u_1, \cdots, u_\nu = 1}^{s} \frac{(n_{u_1 \cdots u_\nu} - n^{1-\nu} n_{u_1} \cdots n_{u_\nu})^2}{n^{1-\nu} n_{u_1} \cdots n_{u_\nu}}. \tag{1.3}$$

In Section 3 we show that under the hypothesis that $\{X_n\}$ is stationary and independent, the distribution function of (1.3) approaches, as $n \to \infty$, the distribution function

$$\mathop{*}_{\lambda=1}^{\nu-1} K_{s^{\nu-1-\lambda}(s-1)^2}(x/\lambda). \tag{1.4}$$

In the special case $\nu = 2$ this result is implicit in the work of Hoel [2]. Note that in this case (1.4) becomes $K_{(s-1)^2}(x)$.

The means and variances of the distributions (1.2) and (1.4) are easily written down. It is obvious that if ν is fixed and $s \to \infty$, then these distributions are, when normed by their means and standard deviations, asymptotically normal. It is a simple matter to show, using Ljapunov's condition and the fact that the distributions are convolutions, that the same thing is true if s is fixed and $\nu \to \infty$. By interpolation in the tables of [3] one can get an approximation to (1.2) for the case $s = 2$ and $\nu = 2$ and an approximation to (1.4) for the case

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$s = 2$ and $\nu = 3$. The paper [3] also deals with the general problem of computing and approximating the distributions of weighted sums of independent chi-square-distributed random variables.

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In his paper "On the serial test for random sequences," forthcoming in this journal, I. J. Good shows that the expected value of the statistic (1.1) and the first moment of its limiting distribution (1.2) have the same value, viz., $s^\nu - 1$.

2. Asymptotic distribution of (1.1). In what follows we make use of the theory of finite dimensional vector spaces. The reader is referred to [4] for the notions of direct sum (donated here by $\mathbf{v}$), spectral decomposition, etc. We denote an operator and the matrix which represents it by the same symbol.

Let $\mathcal{V}_\nu$ be s^ν-dimensional Euclidean space with components indexed by the s^ν ν-tuples $(u_1, \cdots, u_\nu)$ with $1 \leqq u_i \leqq s$. Let

$$\rho_{u_1 \cdots u_\nu} = (p_{u_1} \cdots p_{u_\nu})^{1/2}, \tag{2.1}$$

and let x be the random vector in $\mathcal{V}_\nu$ with components

$$x_{u_1 \cdots u_\nu} = (n_{u_1 \cdots u_\nu} - n\rho^2_{u_1 \cdots u_\nu})/n^{1/2}\rho_{u_1 \cdots u_\nu} .$$

Then $|x|^2$ is the statistic (1.1). Let $M^{(\nu)}$ be the s^ν by s^ν matrix with entries defined by

$$M^{(\nu)}_{u_1 \cdots u_\nu, v_1 \cdots v_\nu} = \delta_{u_1, v_1} \cdots \delta_{u_\nu, v_\nu} - \rho_{u_1 \cdots u_\nu}\rho_{v_1 \cdots v_\nu} .$$

For this matrix to be well defined, the ν-tuples must be ordered. Which order is taken is immaterial so long as it is kept constant throughout the argument.

We show first of all that the covariance matrix of x is asymptotically $\Lambda^{(\nu)}$, where $\Lambda^{(\nu)}$ is the s^ν by s^ν matrix with entries defined by

$$\begin{aligned} \Lambda^{(\nu)}_{u_1 \cdots u_\nu, v_1 \cdots v_\nu} = M^{(\nu)}_{u_1 \cdots u_\nu, v_1 \cdots v_\nu} &+ \sum_{k=1}^{\nu-1} \rho_{u_1 \cdots u_k}\rho_{v_{\nu-k+1} \cdots v_\nu} M^{(\nu-k)}_{u_{k+1} \cdots u_\nu, v_1 \cdots v_{\nu-k}} \\ &+ \sum_{k=1}^{\nu-1} \rho_{u_{\nu-k+1} \cdots u_\nu}\rho_{v_1 \cdots v_k} M^{(\nu-k)}_{u_1 \cdots u_{\nu-k}, v_{k+1} \cdots v_\nu} . \end{aligned} \tag{2.2}$$

Let $\alpha_i[\beta_i]$ be 1 or 0 according as $(X_i, \cdots, X_{i+\nu-1})$ is the ν-tuple $(u_1, \cdots, u_\nu)$ $[(v_1, \cdots, v_\nu)]$ or not. Then $n_{u_1 \cdots u_\nu} = \sum_{i=1}^n \alpha_i$ and $n_{v_1 \cdots v_\nu} = \sum_{i=1}^n \beta_i$. Let $c(i, j) = \text{cov}(\alpha_i, \beta_j)$. Then $c(i, j) = 0$ if $|i - j| \geqq \nu$ and $c(i, i + k)$ is independent of i. Hence, since $|c(i, j)| \leqq 2$, we have

$$\begin{aligned} \text{cov}(n_{u_1 \cdots u_\nu}, n_{v_1 \cdots v_\nu}) &= \textstyle\sum_{i=1}^n \sum_{j=1}^n c(i, j) \\ &= \textstyle\sum_{i=1}^n c(i, i) + \sum_{i=1}^n \sum_{k=1}^{\nu-1} (c(i, i + k) + c(i + k, i)) \\ &\quad - \textstyle\sum_{i=n-\nu+2}^n \sum_{k=n-i+1}^{\nu-1} (c(i, i + k) + c(i + k, i)) \\ &= n[c(1, 1) + \textstyle\sum_{k=1}^{\nu-1} (c(1, 1 + k) + c(1 + k, 1))] + 2\theta\nu^2 \end{aligned}$$

with $|\theta| \leqq 1$. Hence in the limit $\operatorname{cov}(n_{u_1} \cdots_{u_\nu}, n_{v_1} \cdots_{v_\nu})$ is

$$\lim_{n\to\infty} \Lambda^{(\nu)}_{u_1\cdots u_\nu, v_1\cdots v_\nu} =$$

$$\rho^{-1}_{u_1\cdots u_\nu} \rho^{-1}_{v_1\cdots v_\nu}[c(1, 1) + \sum_{k=1}^{\nu-1} (c(1, 1 + k) + c(1 + k, 1))].$$

But for $k = 1, \cdots, \nu - 1$,

$$c(1, 1 + k) = \delta_{u_{k+1}, v_1} \cdots \delta_{u_\nu, v_{\nu-k}} p_{u_1} \cdots p_{u_\nu} p_{v_{\nu-k+1}} \cdots p_{v_\nu} - \rho^2_{u_1\cdots u_\nu} \rho^2_{v_1\cdots v_\nu}.$$

From this expression and similar ones for $c(1, 1)$ and $c(1 + k, 1)$, (2.2) follows.

It is an immediate consequence of the multivariate central limit theorem for ν-dependent random variables [5] that the distribution of x approaches that normal distribution having zero means and having $\Lambda^{(\nu)}$ as covariance matrix.

We proceed now to find the spectral decomposition of $\Lambda^{(\nu)}$. For $\nu > 1$ let $\mathcal{L}_\nu$ be the set of $t \in \mathcal{V}_\nu$ satisfying

$$\sum_{u_1\cdots u_\nu} \rho_{u_1\cdots u_\nu} t_{u_1\cdots u_\nu} = 0 \tag{2.3}$$

and

$$\sum_{u_1} \rho_{u_1} t_{u_1\cdots u_\nu} = \sum_{u_1} \rho_{u_1} t_{u_2\cdots u_\nu u_1}. \tag{2.4}$$

for all $(u_2, \cdots, u_\nu)$. It follows from (2.3) and (2.4) that for $k = 1, \cdots, \nu$ and $i = 1, \cdots, k$,

$$\sum_{u_1\cdots u_k} \rho_{u_1\cdots u_k} t_{u_1\cdots u_\nu} = \sum_{u_1\cdots u_k} \rho_{u_1\cdots u_k} t_{u_{i+1}\cdots u_\nu u_1\cdots u_i}. \tag{2.5}$$

Let $\mathcal{L}_1$ be the set of $t \in \mathcal{V}_1$ for which $\sum_u \rho_u t_u = 0$ and let $\mathcal{L}_0$ consist of the number 0 alone. For $v \geqq 1$, define a linear mapping $\Pi_\nu: \mathcal{L}_\nu \to \mathcal{L}_{\nu-1}$ by $(\Pi_\nu t)_{u_1\cdots u_{\nu-1}} = \sum_u \rho_u t_{uu_1\cdots u_{\nu-1}}$. That $t \in \mathcal{L}_\nu$ implies $\Pi_\nu t \in \mathcal{L}_{\nu-1}$ can be verified by computation. For $\nu \geqq 2$, define a second linear mapping $\Omega_{\nu-1}: \mathcal{L}_{\nu-1} \to \mathcal{L}_\nu$ by

$$(\Omega_{\nu-1} t)_{u_1\cdots u_\nu} = \rho_{u_1} t_{u_2\cdots u_\nu} + \rho_{u_\nu} t_{u_1\cdots u_{\nu-1}} - \rho_{u_1 u_\nu}(\Pi_{\nu-1} t)_{u_2\cdots u_{\nu-1}}. \tag{2.6}$$

If $\nu = 2$ the last term in (2.6) is to be omitted. Again a computation shows that $\Omega_{\nu-1} t \in \mathcal{L}_\nu$ if $t \in \mathcal{L}_{\nu-1}$. From these definitions it follows that

$$\Pi_\nu \Omega_{\nu-1} t = t, \quad t \in \mathcal{L}_{\nu-1}. \tag{2.7}$$

Let $\mathcal{L}^0_\nu$ be the set of $t \in \mathcal{L}_\nu$ such that $\Pi_\nu t = 0$. Then

$$\mathcal{L}_\nu = \mathcal{L}^0_\nu \vee \Omega_{\nu-1}(\mathcal{L}_{\nu-1}). \tag{2.8}$$

In fact if $t \in \mathcal{L}_\nu$, then $t = (t - \Omega_{\nu-1}\Pi_\nu t) + \Omega_{\nu-1}\Pi_\nu t$, while $t - \Omega_{\nu-1}\Pi_\nu t \in \mathcal{L}^0_\nu$ and $\Omega_{\nu-1}\Pi_\nu t \in \Omega_{\nu-1}(\mathcal{L}_{\nu-1})$. And if $t \in \mathcal{L}^0_\nu \cap \Omega_{\nu-1}(\mathcal{L}_{\nu-1})$, then $t = \Omega_{\nu-1}t'$ and $0 = \Pi_\nu\Omega_{\nu-1}t' = t'$ so $t = 0$.

If $t \varepsilon \mathcal{L}_\nu$ then $M^{(\nu)}t = t$. From this and (2.5) it follows that

$$\sum_{v_1 \cdots v_\nu} \rho_{v_{\nu-k+1} \cdots v_\nu} M^{(\nu-k)}_{u_{k+1} \cdots u_\nu, v_1 \cdots v_{\nu-k}} t_{v_1 \cdots v_\nu}$$

$$= \sum_{v_1 \cdots v_{\nu-k}} M^{(\nu-k)}_{u_{k+1} \cdots u_\nu, v_1 \cdots v_{\nu-k}} (\Pi_{\nu-k+1} \cdots \Pi_\nu t)_{v_1 \cdots v_{\nu-k}} = (\Pi_{\nu-k+1} \cdots \Pi_\nu t)_{u_{k+1} \cdots u_\nu}.$$

Using this relation and the symmetric one, we get

$$(2.9) \qquad \begin{aligned} (\Lambda^{(\nu)}t)_{u_1 \cdots u_\nu} = t_{u_1 \cdots u_\nu} &+ \sum_{k=1}^{\nu-1} \rho_{u_1 \cdots u_k} (\Pi_{\nu-k+1} \cdots \Pi_\nu t)_{u_{k+1} \cdots u_\nu} \\ &+ \sum_{k=1}^{\nu-1} \rho_{u_{\nu-k+1} \cdots u_\nu} (\Pi_{\nu-k+1} \cdots \Pi_\nu t)_{u_1 \cdots u_{\nu-k}} \end{aligned}$$

for $t \varepsilon \mathcal{L}_\nu$. If $\nu \geqq 3$ and $t \varepsilon \mathcal{L}_{\nu-1}$, then by (2.7) and (2.9) we have

$$\begin{aligned} ((\Lambda^{(\nu)}\Omega_{\nu-1} - \Omega_{\nu-1})t)_{u_1 \cdots u_\nu} = \rho_{u_1} t_{u_2 \cdots u_\nu} &+ \sum_{k=2}^{\nu-1} \rho_{u_1 \cdots u_k} (\Pi_{\nu-k+1} \cdots \Pi_{\nu-1} t)_{u_{k+1} \cdots u_\nu} \\ + \rho_{u_\nu} t_{u_1 \cdots u_{\nu-1}} &+ \sum_{k=2}^{\nu-1} \rho_{u_{\nu-k+1} \cdots u_\nu} (\Pi_{\nu-k+1} \cdots \Pi_{\nu-1} t)_{u_1 \cdots u_{\nu-k}}. \end{aligned}$$

From this, using (2.9) again, one shows by a long but straightforward calculation that for $v \geqq 3$

$$(2.10) \qquad \Lambda^{(\nu)}\Omega_{\nu-1} - \Omega_{\nu-1} = \Omega_{\nu-1}\Lambda^{(\nu-1)}$$

on $\mathcal{L}_{\nu-1}$.

We next show that for $\nu \geqq 2$

$$(2.11) \qquad \Lambda^{(\nu)} = I + \sum_{k=2}^{\nu} \Omega_{\nu-1} \cdots \Omega_{k-1} \Pi_k \cdots \Pi_\nu$$

on $\mathcal{L}_\nu$. The proof goes by induction. The verification being simple for $\nu = 2$, assume (2.11) holds with ν replaced by $\nu - 1$. Then by (2.10) and (2.7) we have

$$\Lambda^{(\nu)}\Omega_{\nu-1} = \left(I + \sum_{k=2}^{\nu} \Omega_{\nu-1} \cdots \Omega_{k-1} \Pi_k \cdots \Pi_\nu\right) \Omega_{\nu-1}.$$

In other words, it follows that (2.11) holds on $\Omega_{\nu-1}(\mathcal{L}_{\nu-1})$. Since it obviously holds on $\mathcal{L}_\nu^0$, it follows by (2.8) that (2.11) holds on all of $\mathcal{L}_\nu$.

Let $\mathfrak{M}_1 = \mathcal{L}_\nu^0$ and for $\lambda = 2, \cdots, \nu$ let

$$(2.12) \qquad \mathfrak{M}_\lambda = \Omega_{\nu-1} \cdots \Omega_{\nu+1-\lambda} \mathcal{L}^0_{\nu+1-\lambda}.$$

It follows from (2.8) by induction that $\mathcal{L}_\nu = \mathfrak{M}_1 \vee \cdots \vee \mathfrak{M}_\nu$. Using (2.11) one easily shows that for $\lambda = 1, \cdots, \nu$,

$$(2.13) \qquad \Lambda^{(\nu)}t = \lambda t \quad \text{if} \quad t \varepsilon \mathfrak{M}_\lambda.$$

Let $\sigma \varepsilon \mathcal{V}_\nu$ be the vector whose $(u_1, \cdots, u_\nu)$-th component is $\rho_{u_1 \cdots u_\nu}$ and let $\sigma(v_1, \cdots, v_{\nu-1}) \varepsilon \mathcal{V}_\nu$ be the vector whose $(u_1, \cdots, u_\nu)$-th component is

$\rho_{u_1}\delta_{u_2,v_1} \cdots \delta_{u_\nu,v_{\nu-1}} - \rho_{u_\nu}\delta_{u_1,v_1} \cdots \delta_{u_{\nu-1},v_{\nu-1}}$. Let $\mathfrak{M}_0$ be the manifold generated by σ and the $s^{\nu-1}$ vectors $\sigma(v_1, \cdots, v_{\nu-1})$. By definition $\mathfrak{L}_\nu$ is the orthogonal complement of $\mathfrak{M}_0$, so that

$$\mathfrak{V}_\nu = \mathfrak{M}_0 \vee \mathfrak{M}_1 \vee \cdots \vee \mathfrak{M}_\nu . \tag{2.14}$$

Direct computations show that $\Lambda^{(\nu)}\sigma = \Lambda^{(\nu)}\sigma(v_1, \cdots, v_{\nu-1}) = 0$, so that (2.13) holds for $\lambda = 0$. Thus each $\mathfrak{M}_\lambda$, $0 \leqq \lambda \leqq \nu$, consists of eigenvectors with eigenvalue λ. These are all the invariant subspaces, in view of (2.14).

We now compute the dimension of $\mathfrak{L}_\nu^0$. It is easy to show that $\dim \mathfrak{L}_1^0 = s - 1$. Suppose $\nu \geqq 2$. To say that $t \in \mathfrak{L}_\nu^0$ is to say that for all $u_1, \cdots, u_{\nu-1}$

$$\sum_u \rho_u t_{uu_1\cdots u_{\nu-1}} = \sum_u \rho_u t_{u_1\cdots u_{\nu-1}u} = 0. \tag{2.15}$$

Let X be the $s^{\nu-1}$ by s^ν matrix with entries

$$X_{u_1\cdots u_{\nu-1},v_1\cdots v_\nu} = \rho_{v_1}\delta_{u_1,v_2} \cdots \delta_{u_{\nu-1},v_\nu}$$

and let Y be the $s^{\nu-1}$ by s^ν matrix with entries

$$Y_{u_1\cdots u_{\nu-1},v_1\cdots v_\nu} = \rho_{v_\nu}\delta_{u_1,v_1} \cdots \delta_{u_{\nu-1},v_{\nu-1}} .$$

The partitioned matrix

$$Z = \begin{bmatrix} X \\ Y \end{bmatrix}$$

is the matrix of the system (2.15), i.e., t lies in $\mathfrak{L}_\nu^0$ if and only if t is orthogonal to each row of Z. In order to find the (column) rank of Z let A and B be column vectors with $s^{\nu-1}$ components $A_{u_1\cdots u_{\nu-1}}$ and $B_{u_1\cdots u_{\nu-1}}$ respectively, and let C be the partitioned vector

$$C = \begin{bmatrix} A \\ B \end{bmatrix}.$$

Now C is orthogonal to each column of Z if and only if $\rho_{u_1}A_{u_2\cdots u_\nu} = -\rho_{u_\nu}B_{u_1\cdots u_{\nu-1}}$. Thus if for a set $\{D_{u_2\cdots u_{\nu-1}}\}$ of $s^{\nu-2}$ numbers we let $A_{u_2\cdots u_\nu} = \rho_{u_\nu}D_{u_2\cdots u_{\nu-1}}$ and $B_{u_1\cdots u_{\nu-1}} = -\rho_{u_1}D_{u_2\cdots u_{\nu-1}}$, then C is orthogonal to the columns of Z. Conversely, if C is orthogonal to these columns, it can be cast in this form. Hence the subspace of $2s^{\nu-1}$-dimensional space orthogonal to the subspace generated by the columns of Z has dimension $s^{\nu-2}$. Therefore Z has rank $2s^{\nu-1} - s^{\nu-2}$ and $\dim \mathfrak{L}_\nu^0 = s^{\nu-2}(s - 1)^2$. It now follows by (2.12) and (2.14) that $\dim \mathfrak{M}_0 = s^{\nu-1}$, $\dim \mathfrak{M}_\nu = s - 1$, and $\dim \mathfrak{M}_\lambda = s^{\nu-1-\lambda}(s - 1)^2$ for $\lambda = 1, 2, \cdots, \nu - 1$.

We now have the dimensions of the invariant subspaces and hence the multiplicities of the eigenvalues of $\Lambda^{(\nu)}$. Since the distribution of x is asymptotically normal with covariance matrix, $\Lambda^{(\nu)}$, it follows by an obvious generalization of the result of Section 24.5 of [6] that the distribution of $|x|^2$, or (1.1), approaches (1.2), under the null hypothesis.

3. Asymptotic distribution of (1.3). We assume now that $\{X_n\}$ is independent and stationary, but we regard the p_k as unknown. In fact, let $p_1, \cdots, p_{s-1}$ be parameters to be estimated, define p_s by $p_s = 1 - \sum_{k=1}^{s-1} p_k$, and let $\rho^2_{u_1\cdots u_\nu}$ be a function of $p_1, \cdots, p_{s-1}$ defined by (2.1).

Now it is easy to show that the values p_k which maximize

$$\prod_{u_1\cdots u_\nu} (\rho^2_{u_1\cdots u_\nu})^{n_{u_1\cdots u_\nu}}$$

are $p_k = n_k/n + \epsilon_k$, where $\epsilon_k = 0(1/n)$. And now the reasoning of Section 30.3 of [6] becomes applicable. Let B be the s^ν by $s - 1$ matrix with entries $B_{u_1\cdots u_\nu,u} = \rho^{-1}_{u_1\cdots u_\nu}\partial\rho^2_{u_1\cdots u_\nu} / \partial p_u$, $1 \leqq u_i \leqq s$, $1 \leqq u < s$. Let y be the random vector which results from substituting the estimate n_k / n for p_k in x. Then $|y|^2$ is (1.3). In order that the theorem of Section 30.3 of [6] be directly applicable it would be necessary that $\{n_{u_1\cdots u_\nu}\}$ be a sample from a multinomial universe. However, since the x defined above is asymptotically normal with covariance matrix $\Lambda^{(\nu)}$, since B has rank $s - 1$ and since $x_{u_1\cdots u_\nu} = o(n^{1/4})$ in probability (as is easily shown), a perusal of the proof of the theorem referred to shows that we are in the present case justified in concluding that the distribution of y approaches that normal distribution with zero means and covariance matrix $A\Lambda^{(\nu)}A'$, where $A = I - B(B'B)^{-1}B'$.

We now find the spectral decomposition of $A\Lambda^{(\nu)}A'$. Let K be the s by $s - 1$ matrix with entries $K_{u,v}$, where $K_{u,v} = \delta_{u,v}$ if $u < s$ and $K_{s,v} = -1$. Let J be the s^ν by s matrix with entries

$$J_{u_1\cdots u_\nu,u} = \sum_{i=1}^{\nu} \rho_{u_1\cdots u_{i-1}u_{i+1}\cdots u_\nu}\delta_{u_i,u}\rho_u^{-1}.$$

Then $B = JK$.

If $t \,\varepsilon\, \mathfrak{L}_\nu$ it follows from (2.5) that $(J't)_u = \nu\rho_u^{-1}(\Pi_2 \cdots \Pi_\nu t)_u$. From this it follows that $J't = 0$ for $t \,\varepsilon\, \mathfrak{M}_1 \vee \cdots \vee \mathfrak{M}_{\nu-1}$. If $\sigma(v_1, \cdots, v_{\nu-1})$ is defined as in Section 2, then, as a direct computation shows, $J'\sigma(v_1, \cdots, v_{\nu-1}) = 0$. Moreover, $(J'\sigma)_u = \nu$, so that $B'_\sigma = 0$. Hence $B't = 0$ for $t \,\varepsilon\, \mathfrak{M}_0 \vee \cdots \vee \mathfrak{M}_{\nu-1}$. Now the matrix A is symmetric and idempotent, so that, viewed as an operator, it is a perpendicular projection on the manifold $\mathfrak{N} = \{t{:}At = t\} = \{t{:}B(B'B)^{-1}B't = 0\}$. It is easy to show that the rank of $B(B'B)^{-1}B'$ is the same as that of B, viz., $s - 1$. Hence $\dim \mathfrak{N} = s^\nu - s + 1$. We have shown that $\mathfrak{M}_0 \vee \cdots \vee \mathfrak{M}_{\nu-1} \subset \mathfrak{N}$, and since $\dim (\mathfrak{M}_0 \vee \cdots \vee \mathfrak{M}_{\nu-1}) = s^\nu - s + 1$ (cf. Section 2), we have $\mathfrak{M}_0 \vee \cdots \vee \mathfrak{M}_{\nu-1} = \mathfrak{N}$. The manifolds $\mathfrak{M}_\lambda$, being the invariant spaces of the symmetric matrix $\Lambda^{(\nu)}$, are mutually orthogonal. Hence $\mathfrak{M}_\nu$ is the orthogonal complement of $\mathfrak{N}$ and $At = 0$ for $t \,\varepsilon\, \mathfrak{M}_\nu$. Therefore $A\Lambda^{(\nu)}A't = \lambda t$ if $t \,\varepsilon\, \mathfrak{M}_\lambda$ with $1 \leqq \lambda < \nu$, while $A\Lambda^{(\nu)}A't = 0$ if $t \,\varepsilon\, \mathfrak{M}_0 \vee \mathfrak{M}_\nu$. Finally $\dim \mathfrak{M}_\lambda = s^{\nu-1-\lambda}(s - 1)^2$ for $\lambda = 1, \cdots, v - 1$ and $\dim \mathfrak{M}_0 \vee \mathfrak{M}_\nu = s^{\nu-1} + s - 1$.

Thus we have the eigenvalues of $A\Lambda^{(\nu)}A'$, with their multiplicities, and it follows as in Section 2 that the distribution of $|y|^2$, or (1.3), approaches (1.4).

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