and for some $\alpha > 2$ and constant C_2

$$E \mid X_M(t) - EX_M(t) \mid^\alpha \leqq C_2 \qquad \text{for all } t. \tag{3.12}$$

For M large enough, (3.11) follows from (3.1), (3.10) and (3.5). By Minkowski's inequality, (3.12) follows from (3.2) and (2.4). The proof of the theorem is now completed.

4. A remark on applications. One use of the foregoing central limit theorem is to provide conditions, without any further ado, for the asymptotic normality of various estimates of the spectrum of a stationary time series that have been considered by us (see [4]).

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ON THE ENUMERATION OF DECISION PATTERNS INVOLVING n MEANS[1]

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1. Introduction. The purpose of this paper is to provide a mathematical treatment for the enumeration of decision patterns obtained in the pairwise comparison of n sample means. In the comparison of n means, there are altogether $\binom{n}{2}$ pairwise comparisons, and each individual comparison between two means, say m_1 and m_2, must result in the decision that m_1 is significantly less than m_2, that m_2 is significantly less than m_1, or that there is no significant difference. Symbolically, these three alternatives are written as $m_1 < m_2$, $m_2 < m_1$, and $m_1 \doteq m_2$, respectively.

There are, thus, altogether $3^{\binom{n}{2}}$ possible *decision sets* in the comparison of n objects, a *decision set* consisting of the $\binom{n}{2}$ pairwise comparisons. However, for the comparison of n means, there are fewer decision sets since circularities are automatically ruled out.

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A decision set involving n means can be represented symbolically with the use of the following scheme:

If $m_1 < m_2$, the letter m_1 is written to the left of m_2; if $m_1 \doteq m_2$, the letters m_1 and m_2 are underlined with what we shall call an indifference line and it does not matter whether we write $\underline{m_1m_2}$ or $\underline{m_2m_1}$. We shall also write $\underline{m_1m_2m_3}$ to express the fact that $m_1 \doteq m_2$, $m_2 \doteq m_3$, and $m_1 \doteq m_3$. In general, the fact that $m_i \doteq m_j$ will be expressed by an indifference line common to m_i and m_j.

The following are two simple examples illustrating this representation of decision sets:

(i) The decision set $m_1 < m_2$, $m_1 < m_3$, $m_1 < m_4$, $m_2 \doteq m_3$, $m_2 < m_4$, and $m_3 \doteq m_4$ is written as

$$m_1 \quad \underline{m_2 \quad \underline{m_3}}\underline{\quad m_4},$$

and

(ii) the decision set $m_1 \doteq m_2$, $m_1 \doteq m_3$, $m_1 < m_4$, $m_2 \doteq m_3$, $m_2 \doteq m_4$, and $m_3 \doteq m_4$ is written as

$$\underline{m_1 \quad \underline{m_2 \quad m_3}}\underline{\quad m_4}.$$

If the only difference between the schematic presentation of two decision sets is a permutation of the means $m_1, m_2, m_3, \cdots, m_n$, they are said to belong to the same *decision pattern*. A decision pattern is, therefore, characterized by the number of means and the number and the arrangement of the indifference lines. The decision pattern corresponding to a given decision set will be indicated by replacing the mean with dots. The decision pattern corresponding to the first example above is

$$\cdot \quad \underline{\cdot \quad \underline{\cdot}}\underline{\quad \cdot},$$

and that of the second example is

$$\underline{\cdot \quad \underline{\cdot \quad \cdot}}\underline{\quad \cdot}$$

An important point which must be observed in the construction of decision patterns is that no indifference line is completely covered by another indifference line.

Having defined decision patterns and decision sets, one may now ask

(a) What is the total number of distinct decision sets in the pairwise comparison of n means?

(b) What is the total number of distinct decision patterns in the pairwise comparison of n means?

In this paper it will be shown that the number of decision patterns involving n sample means is

$$f(n) = \frac{1}{n+1}\binom{2n}{n}. \tag{1.1}$$

Although question (a) can, of course, be answered by direct enumeration for small values of n, the general problem is as yet unsolved.

2. Derivation of formula for number of decision patterns. In order to derive formulas giving the total number of decision patterns, consider the last k dots on the right in a pattern which has n dots, $n \geqq k$. Beneath each pair of dots, a_i and a_{i+1}, where $i = 1, 2, \cdots, k - 1$, j line segments, $j = 0, 1, 2, \cdots$, may be drawn, each line being part of an indifference line, or a whole one. The step from a_i to a_{i+1}, called the "i-th step," may be made in many ways, as indicated by the number of line segments underlining the pair of dots. Let s_j denote a step with j line segments. It should be noted that each line segment under dots a_i and a_{i+1} may or may not be part of an indifference line including several other dots. A dot a_i is called a "right terminal dot" ("left terminal dot") of an indifference line whenever the indifference line does not extend to $a_{i+1}(a_{i-1})$.

Let $f_j(k)$, $j = 0, 1, 2, \cdots$, denote the total number of decision patterns possible when the first step of k dots is s_j.

It can be seen easily that

$$f_0(k) = f_0(k - 1) + f_1(k - 1), \quad k \geqq 3, \tag{2.1}$$

since the number of decision patterns for k dots with first step s_0 is the same as the sum of the number of decision patterns for $k - 1$ dots with the first steps s_0 and s_1 (k cannot be $\leqq 2$, since $f_1(k - 1)$ would be undefined).

A general recursion formula for $f_e(k)$ with $e = 1, 2, 3, \cdots$ may be written as

$$f_e(k) = f_{e-1}(k - 1) + 2f_e(k - 1) + f_{e+1}(k - 1), \qquad k \geqq 3. \tag{2.2}$$

To prove (2.2), assume that s_e is the first step, in which case it is necessary that $n \geqq k + e - 1$. It must be large enough so that no two of the e indifference lines of the first step have a_1 or any dot to the left of a_1 as a common left terminal dot. At least $e - 1$ of the indifference lines in step one must be continued beyond a_2, since two indifference lines can not have a common right terminal dot at a_2. The second step, thus, has at least $e - 1$ indifference lines. On the other hand, not more than $e + 1$ indifference lines are possible in the second step, since two indifference lines would otherwise have a_2 as a common left terminal dot. Thus, if the first step is s_e and only one of its indifference lines terminates at a_2, the second step is s_{e-1} or s_e; if the first step is such that no indifference lines terminate at a_2, the second step is s_e or s_{e+1}.

For certain values of k, s_e is an impossible first step, and $f_e(k)$ is equal to zero. (It will be assumed here that n is sufficiently large so that not more than one indifference line has a left terminal dot at a_1.) If k is an arbitrary positive integer, say r, then s_{r-1} is a possible first step since each a_i, where $i = 2, 3, \cdots, r$, may be a right terminal dot for exactly one indifference line. If the first step were s_r, then some point a_i would have to be a common right terminal dot for two indifference lines. Thus $f_e(r) = 0$, when $e = r, r + 1, \cdots$, and, in general,

$$f_e(k) = 0, \tag{2.3}$$

where

$$e \geqq k > 1.$$

Let $f(n)$ denote the total number of decision patterns for n means. Clearly,

$$f(n) = f_0(n) + f_1(n), \tag{2.4}$$

since s_0 and s_1 are the only possible first steps.

Since $f(n)$ depends only on $f_0(n)$ and $f_1(n)$, equations (2.1), (2.2), and (2.3), together with the boundary conditions

$$f_0(1) = f_0(2) = f_1(2) = 1, \tag{2.5}$$

will lead to (1.1).

Using standard techniques for solving difference equations, it can be shown that[3]

$$f_e(k) = \frac{2e+1}{e+k}\binom{2k-2}{k+e-1}. \tag{2.6}$$

This result can be verified by substituting (2.6) into equations (2.1), (2.2), (2.3), and (2.5). It follows immediately that

$$f(n) = f_0(n) + f_1(n) = \frac{1}{n+1}\binom{2n}{n}.$$

PERCENTILES OF THE ω_n STATISTIC[1]

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If n points are selected independently from a uniform distribution on a unit interval there arise $n + 1$ subintervals, each of expected length $1/(n + 1)$. If L_k is the length of the kth interval from the left, then

$$\omega_n = \tfrac{1}{2}\sum_{k=1}^{n+1}\left|L_k - \frac{1}{n+1}\right|.$$

The distribution function of ω_n is 0 for $x < 0$, 1 for $x > n/(n + 1)$, and for $0 \leqq x \leqq n/(n + 1)$

$$F_n(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0 + 1,$$

where

$$b_k = \sum_{q=0}^{r} (-1)^{q+k+1}\binom{n+1}{q+1}\binom{q+k}{q}\binom{n}{k}\left(\frac{n-q}{n+1}\right)^{n-k},$$

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