the last equality following from (9). Since $f(x)$ is an arbitrary Ω_ϕ-integrable function, this last equation, being true for all $B \varepsilon \mathfrak{F}_{s_1}$, shows that s_1 is a sufficient statistic for $\{\mu_\theta ; \theta \varepsilon \Omega_\phi\}$. But we are given that t is minimal sufficient for $\{\mu_\theta ; \theta \varepsilon \Omega_\phi\}$. Hence there is a mapping h of S_1 onto T such that $t(x) = h(s_1(x))$, $[\{\mu_\theta t^{-1}\}_\phi]$. If we now restrict x to X_ϕ it is evident that $t(x) = h(s(x))$, $[\{\mu_\theta^\phi t^{-1}\}]$, as was to be proved.

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A CENTRAL LIMIT THEOREM FOR MULTILINEAR STOCHASTIC PROCESSES

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1. Introduction. Let the random sequence $X(t)$ be observed for $t = 1, 2, \cdots$, and let $S(n) = X(1) + \cdots + X(n)$ be its consecutive sums. The random sequence may be said to obey the *classical* central limit theorem if, for any real number a,

$$\lim_{n\to\infty} \text{Prob}\left\{\frac{S(n) - ES(n)}{\sigma[S(n)]} < a\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-1/2x^2}\, dx. \tag{1.1}$$

Because of the importance of the central limit theorem in establishing the properties of statistical tests and estimates, it would appear that in order to develop a satisfactory theory of statistical inference for stochastic processes which are random sequences of dependent random variables, it is necessary to establish a central limit theorem for such processes. Diananda [2] has proved a central limit theorem for discrete parameter stochastic processes which are *linear* processes. We here introduce a class of stochastic processes which we call *multilinear* processes, for which we prove a central limit theorem. The results are capable of extension to the continuous parameter case, but we do not do so here.

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2. Multilinear processes. A random sequence $X(t)$, defined for $t = 0, \pm 1, \cdots$ is said to be a *multilinear* process if it can be represented as follows:

$$X(t) = \sum_{v_1, \cdots, v_K = -\infty}^{\infty} a(v_1, \cdots, v_K) W_1(t - v_1) \cdots W_K(t - v_K), \tag{2.1}$$

where K is a positive integer; the $a(v_1, \cdots, v_K)$ are constants, defined for $v_i = 0, \pm 1, \cdots$ and $i = 1, \cdots, K$, such that

$$\sum_{v_1, \cdots, v_K = -\infty}^{\infty} | a(v_1, \cdots, v_K) | < \infty, \tag{2.2}$$

and the $W_i(t)$ are random variables, defined for $t = 0, \pm 1, \cdots$ and $i = 1, \cdots, K$, such that the K-dimensional random vectors $\mathbf{W}(t) = [W_1(t), \cdots, W_K(t)]$ are independent. For the sake of clarity, we give the definition of independence; the $\mathbf{W}(t)$ are independent if, for any integer n, for any set of points $t_1, \cdots, t_n$, and any set of bounded Borel functions of K-variables $g_1(\mathbf{w}), \cdots, g_n(\mathbf{w})$, it holds that

$$E[g_1(\mathbf{W}(t_1)) \cdots g_n(\mathbf{W}(t_n))] = Eg_1(\mathbf{W}(t_1)) \cdots Eg_n(\mathbf{W}(t_n)). \tag{2.3}$$

It is also assumed that the random variables $W_i(t)$ satisfy the condition that, for some $\alpha > 2$ and constant C,

$$E \mid W_1(t_1) \cdots W_K(t_K) \mid^{\alpha} \leqq C \qquad \text{for any } t_1, \cdots, t_K. \tag{2.4}$$

Random sequences which admit a representation of the form of (2.1), with $K = 1$, have been called by Bartlett ([1], p. 146) linear processes. The interest of multilinear processes derives from the fact that they possess certain closure properties, if it is assumed that the random variables $W_i(t)$ which occur in the definition of the multilinear processes possess moments of sufficiently high order.

By a linear filter is meant a linear transformation on the space of bounded doubly infinite sequences $\{X(t), t = 0, \pm 1, \cdots\}$ to itself, of the form

$$Y(t) = \sum_{v=-\infty}^{\infty} k(v) X(t - v), \tag{2.5}$$

where

$$\sum_{v=-\infty}^{\infty} | k(v) | < \infty. \tag{2.6}$$

Multilinear processes are closed under the operation of linear filtering, in the sense that if $X(t)$ is a multilinear process, then so is $Y(t)$.

It may be verified that powers of multilinear processes are multilinear processes. More generally, polynomials in multilinear processes

$$P[X(t)] = c_n X^n(t) + \cdots + c_1 X(t) + c_0$$

are multilinear processes.

Next, we note that the sum and product of the multilinear process $X(t)$, and a different multilinear process

$$X'(t) = \sum a'(v_1, \cdots, v_{K'}) W_1'(t - v_1) \cdots W_{K'}'(t - v_{K'}),$$

is again a multilinear process if the $(K + K')$ dimensional random vectors $[W_1(t), \cdots, W_K(t), W_1'(t), \cdots, W_{K'}'(t)]$ form an independent sequence. Since the process defined by $X'(t) = X(t + \tau)$, where τ is fixed, satisfies this condition, it follows that $X(t)X(t + \tau)$ is a multilinear process.

The model of a multilinear process appears to be applicable to many stochastic processes which are physically observed. For many physical processes may be regarded as arising from independent random variables through a finite bank of linear filters and non-linear power law interactions which represent on the one hand the effect of measuring devices, and on the other hand the transmission properties of nature from the regions where the independent events occurred to where they (or their superpositions) are observed.

3. A central limit theorem.

THEOREM. *Let $X(t)$ be a multilinear process, in the sense that it admits of a representation of the form of* (2.1), *with* (2.2), (2.3) *and* (2.4) *all being true. Then* (1.1) *holds if*

$$\liminf_n \frac{1}{n} \sigma^2[S(n)] > 0. \tag{3.1}$$

PROOF: For any positive integer M, let V_M be the set of K-tuples $(v_1, \cdots, v_K)$ whose components v_i are integers and satisfy $|v_i| \leqq M$. Define, for $t = 0, \pm 1, \cdots$

$$X_M(t) = \sum_{(v_1, \cdots, v_K) \varepsilon V_M} a(v_1, \cdots v_K) W_1(t - v_1) \cdots W_K(t - v_K). \tag{3.2}$$

Define the consecutive sums

$$S_M(n) = X_M(1) + \cdots + X_M(n) \tag{3.3}$$

and let $R_M(n) = S(n) - S_M(n)$. To prove the theorem we use the method of iterated probability limits introduced by Marsaglia [3]. We will show that

(i) for every M greater than some M_0, (1.1) is satisfied by the $X_M(t)$; that is, the random sequence $(S_M(n) - ES_M(n))/\sigma[S_M(n)]$ is asymptotically normal with mean 0 and variance 1.

$$\lim_M \limsup_n \frac{\sigma[R_M(n)]}{\sigma[S(n)]} = 0. \tag{ii}$$

$$\lim_M \limsup_n \left| 1 - \frac{\sigma[S_M(n)]}{\sigma[S(n)]} \right| = 0. \tag{iii}$$

In view of Theorems 1 and 2 of Marsaglia [3] the validity of these facts imply the validity of the theorem. To prove (ii) and (iii), we use the inequalities, for any random variables X and Y,

$$\sigma[X + Y] \leqq \sigma[X] + \sigma[Y], \tag{3.4}$$

$$|\sigma[X] - \sigma[Y]| \leqq \sigma[X - Y]. \tag{3.5}$$

From (3.5) it follows that (ii) implies (iii). To establish (ii), we write

$$R_M(n) = \sum_{(v_1, \cdots, v_K) \in V_M} a(v_1, \cdots, v_K) \sum_{t=1}^{n} W_1(t - v_K) \cdots W_K(t - v_K).$$

Consequently, by (3.4),

$$\begin{aligned} \frac{1}{\sqrt{n}} \sigma[R_M(n)] \leqq & \sum_{(v_1, \cdots, v_K) \in V_M} |a(v_1, \cdots, v_K)| \\ & \times \sigma\left[\frac{1}{\sqrt{n}} \sum_{t=1}^{n} W_1(t - v_1) \cdots W_K(t - v_K)\right]. \end{aligned} \tag{3.6}$$

Now

$$\begin{aligned} \sigma^2 & \left[\frac{1}{\sqrt{n}} \sum_{t=1}^{n} W_1(t - v_1) \cdots W_K(t - v_K)\right] \\ & = \frac{1}{n} \sum_{t=1}^{n} \sigma^2[W_1(t - v_1) \cdots W_K(t - v_K)] \\ & + \frac{2}{n} \sum_{s=1}^{n} \sum_{t=s+1}^{n} \operatorname{cov}[W_1(s - v_1) \cdots W_K(s - v_K), W_1(t - v_1) \cdots W_K(t - v_K)]. \end{aligned} \tag{3.7}$$

For fixed $s, v_1, \cdots, v_K$, the covariance in (3.7) vanishes for all t except perhaps for t such that $t = s + v_i - v_j$ for some $i, j = 1, \cdots, K$; there are at most K^2 such values of t.

Now, in view of (2.4), there is a number C_1 such that

$$\sigma^2[W_1(t_1) \cdots W_K(t_K)] \leqq C_1 \qquad \text{for all } t_1, \cdots, t_K. \tag{3.8}$$

Consequently, the variance on the left-hand side of (3.7) is less than $C_1^2[1 + 2K^2]$, which is less than $4C_1^2K^2$. Further, from (3.1), it follows that there is a positive constant B^2 such that for all n,

$$\frac{\sigma^2[S(n)]}{n} \geqq B^2. \tag{3.9}$$

Therefore

$$\frac{\sigma[R_M(n)]}{\sigma[S(n)]} \leqq \frac{1}{B} \frac{\sigma[R_M(n)]}{\sqrt{n}} \leqq \frac{2CK}{B} \sum_{(v_1, \cdots v_K) \in V_M} |a(v_1, \cdots, v_K)|. \tag{3.10}$$

From (3.10) and (2.2) one may infer (ii).

Next, to show (i), we note that the $X_M(t)$ form an $2M$-dependent sequence of random variables. From Marsaglia [3], it follows that a sufficient condition for the $X_M(t)$ to obey the central limit theorem, and thus for (i) to be established, is that for some positive constant B_1,

$$\frac{\sigma^2[S_M(n)]}{n} \geqq B_1^2 \tag{3.11}$$

and for some $\alpha > 2$ and constant C_2

$$E \,|\, X_M(t) - EX_M(t) \,|^{\alpha} \leqq C_2 \qquad \text{for all } t. \tag{3.12}$$

For M large enough, (3.11) follows from (3.1), (3.10) and (3.5). By Minkowski's inequality, (3.12) follows from (3.2) and (2.4). The proof of the theorem is now completed.

4. A remark on applications. One use of the foregoing central limit theorem is to provide conditions, without any further ado, for the asymptotic normality of various estimates of the spectrum of a stationary time series that have been considered by us (see [4]).

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ON THE ENUMERATION OF DECISION PATTERNS INVOLVING n MEANS[1]

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1. Introduction. The purpose of this paper is to provide a mathematical treatment for the enumeration of decision patterns obtained in the pairwise comparison of n sample means. In the comparison of n means, there are altogether $\binom{n}{2}$ pairwise comparisons, and each individual comparison between two means, say m_1 and m_2, must result in the decision that m_1 is significantly less than m_2, that m_2 is significantly less than m_1, or that there is no significant difference. Symbolically, these three alternatives are written as $m_1 < m_2$, $m_2 < m_1$, and $m_1 \doteq m_2$, respectively.

There are, thus, altogether $3^{\binom{n}{2}}$ possible *decision sets* in the comparison of n objects, a *decision set* consisting of the $\binom{n}{2}$ pairwise comparisons. However, for the comparison of n means, there are fewer decision sets since circularities are automatically ruled out.

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